MTH 320 Section 004 Midterm 2 November 18, 2019

1. (10 pts) Does there exist a continuous function f defined on $(0, 1)$, such that

$$\{f(x) : x \in (0, 1)\} = (-1, 0) \cup (1, 2)?$$

Justify your answer.

Solution. Such f does not exist. In fact, for every continuous function f defined on $(0, 1)$, by a corollary of Intermediate Value Theorem, $\{f(x) : x \in (0, 1)\}$ is an interval. But $(-1, 0) \cup (1, 2)$ is not an interval, which is a contradiction.

We may also prove the statement directly using Intermediate Value Theorem. From $\{f(x) : x \in (0, 1)\} = (-1, 0) \cup (1, 2)$ we have $x_1, x_2 \in (0, 1)$ such that $f(x_1) = -0.5$ and $f(x_2) = 1.5$. Applying Intermediate Value Theorem to f on $[x_1, x_2]$ or $[x_2, x_1]$ (depending on $x_1 < x_2$ or $x_2 < x_1$, we conclude that there is some x between x_1 and x_2 such that $f(x) = 0.5$. Now $x \in (0, 1)$ but $f(x) \notin (-1, 0) \cup (1, 2)$, a contradiction. □

2. (10 pts) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers that satisfy $|x_{n+1} - x_n| \le (\frac{1}{2})^n$ for all $n \in \mathbb{N}$. Prove that the sequence (x_n) converges.
Hint: Consider $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$, observe its partial sums, and use comparison test.

Proof. Consider the series $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$. Note that its partial sum sequence is

$$s_n = \sum_{k=1}^{n}(x_{k+1} - x_k) = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_{n+1} - x_n) = x_{n+1} - x_0, \quad n \in \mathbb{N}.$$

So the series $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ converges iff the sequence $(x_{n+1} - x_n)$ converges, which is equivalent to that (x_n) converges. Since $0 < \frac{1}{2} < 1$, we know that $\sum_{n=1}^{\infty}(\frac{1}{2})^n$ converges. Since $|x_{n+1} - x_n| \le (\frac{1}{2})^n$ for all $n \in \mathbb{N}$, by comparison test, $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ converges. Thus, (x_n) converges. □

3. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R.

(a) (3 pts) Write down the formula for R in terms of a_n's.

(b) (3 pts) What is the radius of convergence of $\sum_{n=0}^{\infty} |a_n| x^n$?

(c) (4 pts) Suppose $\sum_{n=0}^{\infty} a_n(-\pi)^n$ converges. Prove that $\sum_{n=0}^{\infty} |a_n| 2^n$ also converges.

Solution. (a) We have

$$R = \frac{1}{\limsup |a_n|^{1/n}}.$$

Here if $\limsup |a_n|^{1/n} = 0$ then $R = \infty$; and if $\limsup |a_n|^{1/n} = \infty$ then $R = 0$.

(b) Let R' denote the radius of convergence of $\sum_{n=0}^{\infty} |a_n| x^n$. Then

$$R' = \frac{1}{\limsup ||a_n||^{1/n}} = \frac{1}{\limsup |a_n|^{1/n}} = R.$$

(b) From that $\sum_{n=0}^{\infty} a_n(-\pi)^n$ converges we then know that $R \geq |-\pi| = \pi$ because if $|-\pi| > R$, the series $\sum a_n x^n$ diverges at $x = -\pi$. Since $\sum |a_n| x^n$ also has radius R, and and $|2| = 2 < \pi \leq R$, we know that $\sum_{n=0}^{\infty} |a_n| 2^n$ converges. □

4. Consider the sequence of functions (f_n) defined by $f_n(x) = \frac{nx}{1+nx}$ on $[0, \infty)$.

   (a) (2 pts) What is the pointwise limit of (f_n) on $[0, \infty)$?

   (b) (4 pts) Does (f_n) converge uniformly on $[0, 1]$?

   (c) (4 pts) Does (f_n) converge uniformly on $[1, \infty)$?

   Justify your answers.

Solution. (a) For $x = 0$, $f_n(x) = 0$ for all n. For $x > 0$, $f_n(x) = \frac{x}{1/n + x} \to \frac{x}{x} = 1$ as $n \to \infty$. Thus, the pointwise limit of (f_n) on $[0, \infty)$ is the function f with $f(0) = 0$ and $f(x) = 1$ for $x > 0$.

(b) (f_n) does not converge uniformly on $[0, 1]$. If it converges uniformly on $[0, 1]$, then since each f_n is continuous on $[0, 1]$, the limit function should be continuous on $[0, 1]$. However, the limit function must be the function f in (a), which is not continuous at 0.

(c) We estimate that for $x \in [1, \infty)$,

$$|f_n(x) - f(x)| = \left| \frac{nx}{1 + nx} - 1 \right| = \frac{1}{1 + nx} \leq \frac{1}{n}.$$

Thus, $\sup\{|f_n(x) - f(x)| : x \in [0, \infty)\} \leq \frac{1}{n} \to 0$ as $n \to \infty$. By a remark in the book, we see that (f_n) converges to f uniformly on $[0, \infty)$. □

5. Let f and g be defined and differentiable on an open interval I.

   (a) (4 pts) Write down the product rule for derivatives.

   (b) (3 pts) Suppose now f and g are both twice differentiable on I. Recall that this means that f' and g' are differentiable on I, and their derivatives are denoted by f'' and g''. Prove that fg is also twice differentiable on I, and express $(fg)''(x)$ for $x \in I$ in terms of $f(x), f'(x), f''(x)$ and $g(x), g'(x), g''(x)$.

   (c) (3 pts) Suppose further that f'' and g'' are both differentiable on I. Then we say that f and g are three times differentiable on I, and denote the derivatives of f'' and g'' by f''' and g''. Prove that fg is also three times differentiable on I, and express $(fg)'''(x)$ for $x \in I$ in terms of $f(x), f'(x), f''(x), f'''(x)$ and $g(x), g'(x), g''(x), g'''(x)$.

Solution. (a) The product rule states that fg is differentiable on I, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

(b) Since f and g are twice differentiable on I, f', g, f, g' are all differentiable on I. By product rule and sum rule, $(fg)' = f'g + fg'$ is differentiable on I and

$$\begin{aligned}(fg)''(x) &= f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x)\\ &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x).\end{aligned}$$

(c) Since f and g are three times differentiable on I, f'', f', f and g'', g', g are all differentiable on I. By product rule and sum rule, $(fg)'' = f''g + 2f'g' + fg''$ is differentiable on I and

$$\begin{aligned}(fg)'''(x) &= f'''(x)g(x)+f''(x)g'(x)+2f''(x)g'(x)+2f'(x)g''(x)+f'(x)g''(x)+f(x)g'''(x)\\ &= f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x).\end{aligned}$$

□