Practice Final, Sec13

Multiple Choice Problems.

1. (S17) Suppose $f(x)$ is a continuous function with values given by the table below.

x	-2	-1	0	1
f(x)	0	3	0	-3

Which of the following statement is correct?

A $f(x) = 2$ has a root $c \in (-1, 0)$.

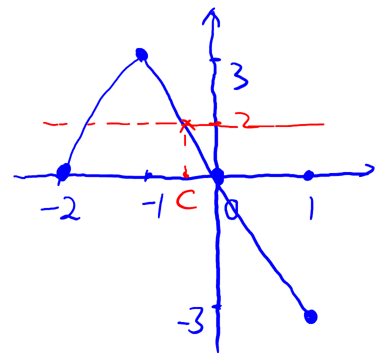
B $f(x) = 2$ has a root $c \in (0, 1)$.

C $f(x) = 4$ has a root $c \in (-1, 0)$.

D $f(x) = 4$ has a root $c \in (-2, 1)$.

E None of the above

$\left.\begin{array}{l} f(-1)=3>2 \\ f(0)=0<2 \end{array}\right\} \xrightarrow{IVT} c\in(-1,0)$, $f(c)=2$.



2. (S17) Suppose $f(x)$ is a differntiable function with values given by the table below.

x	-2	-1	0	1
f(x)	0	3	0	-3

According to Mean Value Theorem, which of the following statement is correct?

A There is $c \in (-1, 0)$ such that $f'(c) = 3$.

B There is $c \in (-2, 0)$ such that $f'(c) = 3$.

C There is $c \in (-1, 1)$ such that $f'(c) = -1$.

D There is $c \in (-2, 1)$ such that $f'(c) = -1$.

E None of the above

$f'(c) = \frac{f(1)-f(-2)}{1-(-2)} = \frac{-3-0}{3} = -1.$

3. Using a linearization at $a = 100$, the linear approximation of $\sqrt{99}$ is

A $\frac{199}{20}$

B $\frac{201}{20}$

C $\frac{99}{10}$

D $\frac{101}{10}$

E None of the above

$f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2}\cdot x^{-\frac{1}{2}}$

$f(100)=\sqrt{100}=10$, $f'(100)=\frac{1}{2}\cdot\frac{1}{\sqrt{100}}=\frac{1}{20}$

$L(x)=f(100)+f'(100)(x-100)=10+\frac{1}{20}(x-100)$

$x=99$. $L(99)=10+\frac{1}{20}(99-100)=10-\frac{1}{20}=\frac{199}{20}$

4. (S17) Suppose you are estimating the root of $x^5 = 33$ using Newton's method. If you use $x_1 = 1$, find the exact value of x_2

A $x_2 = 1 - \frac{32}{5}$

B $x_2 = 1 + \frac{32}{5}$

C $x_2 = 33 - \frac{1}{5}$

D $x_2 = 33 + \frac{32}{5}$

E $x_2 = 1 + \frac{1}{5}$

$\Leftrightarrow x^5-33=0$. $f(x)=x^5-33$

$f'(x)=5\cdot x^4$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\quad x_1=1$, $f(x_1)=1^5-33=-32$

$f'(x_1)=5\cdot 1^4=5$

$= 1 - \frac{-32}{5} = 1+\frac{32}{5}$

5. Evaluate the limit:

$$\lim_{x\to -3^+} \frac{x-2}{x^2(x+3)}$$

A $+\infty$

B $-\infty$

C -5

D 5

E -3

$x \to -3^+$ x approaches -3 from the right

$x-2 \to -3-2=-5$.

$x^2 \to (-3)^2 = 9$

$x+3 \to 0^+$

$\Rightarrow \frac{-5}{9\cdot 0^+} = -\infty$.

6. Find the limit:

$$\lim_{x\to\infty} \frac{x \cancel{-2}}{3x \cancel{+5}}$$

A $+\infty$

B 0

C $\frac{1}{3}$

D $-\frac{2}{3}$

E $-\frac{2}{5}$

highest term rule $= \lim_{x\to\infty} \frac{x}{3x} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3}$.

7. Compute the limit:

$$\lim_{h\to 0} \frac{\frac{1}{h+2}-\frac{1}{2}}{h}$$

A $+\infty$

B $\frac{1}{2}$

C $\frac{1}{4}$

D $-\frac{1}{4}$

E 0

Solution 1: Definition of $f'(2)$. $f(x)=\frac{1}{x}$. $\Rightarrow f'(x) = \frac{-1}{x^2}$

$f'(2) = \lim_{h\to\infty} \frac{\frac{1}{2+h}-\frac{1}{2}}{h}$ $\Rightarrow f'(2) = \frac{-1}{4}$.

Solution 2: $\frac{\frac{1}{h+2}-\frac{1}{2}}{h} = \frac{\frac{2-(2+h)}{(h+2)\cdot 2}}{h} = \frac{\frac{-h}{(h+2)\cdot 2}}{h} = \frac{-1}{(h+2)\cdot 2}$

$\lim_{h\to 0} \frac{-1}{(h+2)\cdot 2} = \frac{-1}{2\cdot 2} = -\frac{1}{4}$.

8. (Spring 16) Find the limit:

$$\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1}$$

$\lim_{\blacksquare\to 0} \frac{\sin \blacksquare}{\blacksquare} = 1$

A -1

B 0

C $\frac{1}{2}$

D $-\frac{1}{1}$

E Does not exist.

$\frac{\sin(x-1)}{x^2-1} = \boxed{\frac{\sin(x-1)}{x-1}} \cdot \frac{x-1}{x^2-1}$

$\lim_{x\to 1} \frac{\sin(x-1)}{x-1} = 1$.

$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

$\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1} = 1 \cdot \frac{1}{2}$.

9. Suppose $\int_0^2 f(x)\ dx = -4$, $\int_0^5 f(x)\ dx = 6$. Find $\int_2^5 f(x)dx$ and the average of $f(x)$ over $[2, 5]$

A $\int_2^5 f(x)dx = 2$, average of f is $\frac{2}{3}$

B $\int_2^5 f(x)dx = 10$, average of f is $\frac{10}{3}$

C $\int_2^5 f(x)dx = -10$, average of f is $-\frac{10}{3}$

D $\int_2^5 f(x)dx = -2$, average of f is $-\frac{2}{3}$

E $\int_2^5 f(x)dx = 10$, average of f is $\frac{10}{5}$

$$\int_2^5 f(x)dx = \int_2^0 f(x)\,dx + \int_0^5 f(x)\,dx$$
$$= -\int_0^2 f(x)\,dx + \int_0^5 f(x)\,dx$$
$$= -(-4) + 6 = 10$$
$$f_{ave} = \frac{1}{5-2}\cdot\int_2^5 f(x)\,dx = \frac{1}{3}\cdot 10$$

10. Evaluate

$$\int_{-\pi}^{\pi} \sin x \cdot \sqrt{\cos x + 2}\ dx$$

A $\frac{4}{3}$

B 0

C $-\frac{4}{3}$

D $-\frac{2}{3}$

E 2

Solution 1:

$f(x) = \sin x \cdot \sqrt{\cos x + 2}$ is odd since $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$

$\Rightarrow f(-x) = -\sin x \cdot \sqrt{\cos x + 2} = -f(x)$.

$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = 0$.

Solution 2: u-sub: $u = \cos x + 2$, $du = -\sin x \cdot dx$, $\cos\pi = -1$, $\cos(-\pi) = -1$.

$\int_{-\pi}^{\pi} \sqrt{\cos x + 2}\cdot \sin x\,dx = \int_1^1 \sqrt{u}\cdot(-du) = 0$.

11. Suppose

$$F(x) = \int_{-\pi}^{\tan x} \sqrt{2 + t^2}\ dt$$

$\left(\int_a^{u(x)} f(t)\,dt\right)' = f(u(x))\cdot u'(x)$

Find $F'(x)$

$u = \tan x$, $u'(x) = \sec^2 x$

$F'(x) = \sqrt{2 + (\tan x)^2}\cdot \sec^2 x$.

A $\sqrt{2 + (\tan x)^2}$

B $\sqrt{2 + (\tan x)^2} \cdot \sec^2 x$

C $\sqrt{2 + t^2} \cdot \sec^2 x$

D $-\sqrt{2 + \pi^2} \cdot \sec^2 x$

E $\int_{-\pi}^{\tan x} \sqrt{2 + (\tan x)^2} \cdot \sec^2 x\ dt$

12. Suppose

$$F(x) = \sqrt{2 + (\tan x)^2}$$

Triple chain rule:

$[f(g(h(x)))]' = f'(g(h))\cdot g'(h)\cdot h'(x)$.

Find $F'(x)$

$f = \sqrt{\square}$, $g = 2 + \bigcirc^2$, $h = \tan x$.

A $\frac{1}{2}(2 + (\tan x)^2)^{-1/2}$

B $\frac{1}{2}(2 + (\tan x)^2)^{-1/2} \cdot (2 \tan x)$

C $\frac{1}{2}(2 + (\tan x)^2)^{-1/2} \cdot (2 \tan x) \cdot (\sec^2 x)$

D $\sqrt{2 + (\tan x)^2} \cdot (2 \tan x) \cdot (\sec^2 x)$

E $\sqrt{2 + (\tan x)^2} \cdot (2 \tan x)$

$$F'(x) = \frac{1}{2}\cdot\square^{-\frac{1}{2}}(2\bigcirc)\cdot(\tan x)'$$
$$= \frac{1}{2}(2 + (\tan x)^2)^{-\frac{1}{2}}\cdot(2\cdot\tan x)\cdot(\sec^2 x)$$

Standard Response Problems.

1. (S17) Calculate the derivatives of $f(x) = x\sin(3x)$. And find the equation of the tangent line to the curve $y = f(x)$ at $x = \frac{\pi}{3}$

product rule: $f'(x) = (x\cdot\sin(3x))' = x'\cdot\sin(3x) + x\cdot(\sin(3x))'$

$= \sin(3x) + x\cdot\cos(3x)\cdot 3$ chain rule.

$x=\frac{\pi}{3}$

slope $= f'(\frac{\pi}{3}) = \sin(\pi) + \frac{\pi}{3}\cdot\cos(\pi)\cdot 3 = 0 + \frac{\pi}{3}\cdot(-1)\cdot 3 = -\pi$, $\sin\pi = 0$, $\cos\pi = -1$.

point: $x = \frac{\pi}{3}$, $y = f(\frac{\pi}{3}) = \frac{\pi}{3}\cdot\sin\pi = \frac{\pi}{3}\cdot 0 = 0$.

Point-Slope formula for tangent line:

$$y = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})\cdot(x-\tfrac{\pi}{3}) = 0 - \pi\cdot(x-\tfrac{\pi}{3}) = \boxed{-\pi(x-\tfrac{\pi}{3})}$$

2. (S17) Suppose $f(x) = \frac{1}{x+7}$

(a) Use the definition of the derivative to find $f'(x)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h+7}-\frac{1}{x+7}}{h} = \lim_{h\to 0}\frac{\frac{x+7-(x+h+7)}{(x+h+7)\cdot(x+7)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{-h}{(x+h+7)(x+7)}}{h}$$

$$= \lim_{h\to 0}\frac{-1}{(x+h+7)\cdot(x+7)}$$

$$= \frac{-1}{(x+7)(x+7)} = \boxed{\frac{-1}{(x+7)^2}}$$

(b) Find the equation of the tangent line to the curve $y = f(x)$ at $x = -2$

Slope $= f'(-2) = \frac{-1}{(-2+7)^2} = -\frac{1}{25}$.

Point: $x = -2$, $y = f(-2) = \frac{1}{-2+7} = \frac{1}{5}$.

Tangent line: $y = \frac{1}{5} - \frac{1}{25}(x-(-2)) = \boxed{\frac{1}{5} - \frac{1}{25}\cdot(x+2)}$

3. (S17) Suppose that y and x satisfy the implicit equation

$$xy^3 + xy = 20$$

(a) Find $\frac{dy}{dx} = y'$ $\quad (xy^3 + xy)' = (20)' \iff (xy^3)' + (x\cdot y)' = 0$

$(x\cdot y^3)' = x'\cdot y^3 + x\cdot(y^3)' = 1\cdot y^3 + x\cdot 3y^2\cdot y'$

$(x\cdot y)' = x'\cdot y + x\cdot y' = y + x\cdot y'$

$\Rightarrow y^3 + \underline{x\cdot 3y^2\cdot y'} + y + \underline{x\cdot y'} = 0$

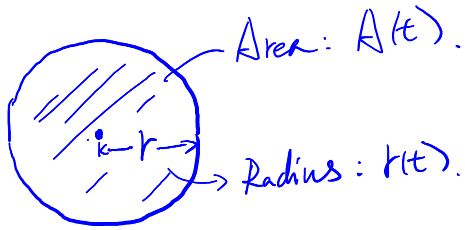
$\Rightarrow (x\cdot 3y^2 + x)\cdot y' = -y^3 - y \Rightarrow \frac{dy}{dx} = y' = \boxed{\frac{-y^3-y}{x\cdot 3y^2+x}}$

(b) Use your answer in part (a) to find the equation of the tangent line to the curve $xy^3 + xy = 20$ at the point $(10, 1)$. $\Rightarrow x=10,\ y=1.$

$$\text{slope} = \frac{dy}{dx} = \frac{-y^3-y}{x\cdot 3y^2+x} = \frac{-1^3-1}{10\cdot 3\cdot 1^2+10} = \frac{-2}{40} = -\frac{1}{20}.$$

Tangent line: $\boxed{y = 1 + \left(-\frac{1}{20}\right)\cdot(x-10)}$

4. If the radius of a circular ink blot is growing at a rate of 3 cm/min. How fast (in cm^2/min) is the area of the blot growing when the radius is 10 cm?



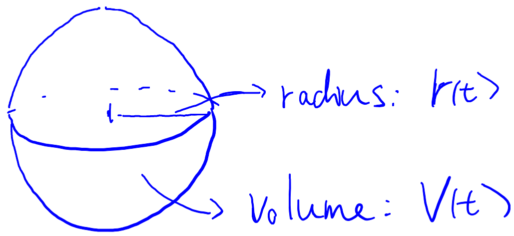
$A = \pi\cdot r^2$

$A' = (\pi\cdot r^2)' = \pi\cdot 2r\cdot r'$

$r' = 3,\ r = 10$

$\Rightarrow A' = \pi\cdot 2\cdot 10\cdot 3 = \boxed{60\pi \text{ cm}^2/\text{min}.}$

5. Air is being pumped into a spherical balloon so that its volume increase at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50cm?



$$V = \frac{4}{3} \cdot \pi \cdot r^3$$

Take derivative (with respect to t).

$$V' = \left(\frac{4}{3}\pi \cdot r^3\right)' = \frac{4}{3}\pi \cdot 3r^2 \cdot r' \quad (*)$$

$$V' = 100, \quad \text{diameter} = 50 \Rightarrow r = 25$$

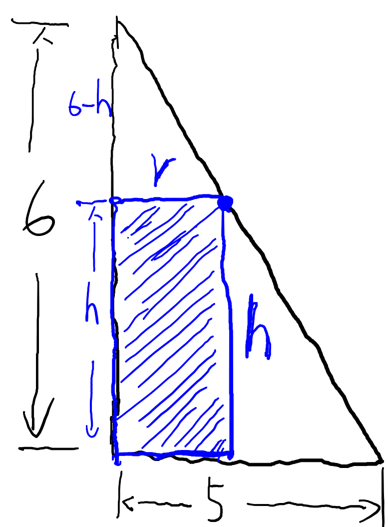
Plug into (*), $100 = \frac{4}{3} \cdot \pi \cdot 3 \cdot (25)^2 \cdot r'$

$$\Rightarrow r' = \boxed{\frac{100}{\frac{4}{3}\pi \cdot 3 \cdot (25)^2} \ cm/s}$$

6. Give a right triangle as below with base 5 cm and height 6 cm. A rectangle is inscribed with its two edges on the right triangle and its upper right corner on the hypotenuse of the right triangle. What are the dimensions of such a rectangle with the greatest possible area?

Area: $A = r \cdot h$ (Target function)

Similar triangles: $\frac{r}{5} = \frac{6-h}{6} \Rightarrow r = \frac{5(6-h)}{6}$



$$A = r \cdot h = \frac{5 \cdot (6-h)}{6} \cdot h = \frac{30-5h}{6} \cdot h = \frac{30h - 5h^2}{6}$$

$$= 5h - \frac{5}{6}h^2$$

$0 \le h \le 6$.

$$A' = \left(5h - \frac{5}{6}h^2\right)' = 5 - \frac{5}{6} \cdot 2h = 0$$

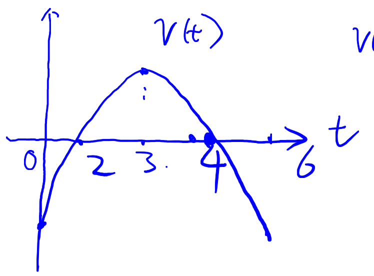
$$\Rightarrow 5 = \frac{5}{3}h \Rightarrow h = 3.$$

$0 \le h \le 3$, $A' > 0$, $3 < h \le 6$, $A' < 0 \Rightarrow A$ attains its max at $h = 3$.

(or. $A(3) = 5 \cdot 3 - \frac{5}{6} \cdot 3^2 = 15 - \frac{15}{2} = \frac{15}{2}$, endpoints: $h=0, h=6$. $A(0) = 0$, $A(6) = 0 \Rightarrow$ A attains max at $h = 3$)

Dimensions: $\boxed{h = 3 \text{ cm}}$, $r = \frac{5(6-h)}{6} = \frac{5 \cdot (6-3)}{6} = \boxed{\frac{5}{2} \text{ cm}}$.

7. A particle moves with velocity $v(t) = -t^2 + 6t - 8$, $0 \le t \le 6$. Sketch the graph of $v(t)$ on $[0, 6]$. When is the acceleration $a(t)$ positive? When does the particle speed up?



$v(t) = -t^2 + 6t - 8 = 0 \Rightarrow t = 2,\ t = 4.$

$a(t) = v'(t) = -2t + 6 > 0 \Rightarrow 0 < t < 3.$

Speed up: $t \in [2,3] \cup [4,6]$.

8. (S16) Suppose $f(x) = x^4 - 6x^2 - 3$.

(a) Identify the intervals over which $f(x)$ is increasing and decreasing, and all values of x where $f(x)$ attains its local maximum or minimum.

$f'(x) = 4x^3 - 12x = 4x\cdot(x^2 - 3) = 4x\cdot(x+\sqrt{3})(x-\sqrt{3}) = 0$

$\Rightarrow x = 0,\quad x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}.$

f': $- - -$ | $+ + +$ | $- - -$ | $+ + +$ on the x-axis, split at $-\sqrt{3}$, 0, $\sqrt{3}$

$f'(-2) < 0 \quad f'(-1) > 0 \quad f'(1) < 0 \quad f'(2) > 0$

Increasing: $[-\sqrt{3}, 0] \cup [\sqrt{3}, +\infty)$

Decreasing: $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$

local max at $x = 0$

local min at $x = -\sqrt{3}$, $x = \sqrt{3}$.

(b) Identify the intervals over which $f(x)$ is concave up and down, and all values of x where $f(x)$ has an inflection point.

$f''(x) = (4x^3 - 12x)' = 12x^2 - 12 = 12(x^2 - 1) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

f'': $+ + +$ | $- - -$ | $+ + +$, split at -1, 1

$f''(-2) > 0 \quad f''(0) < 0 \quad f''(2) > 0$

Concave up: $(-\infty, -1) \cup (1, +\infty)$
$(f'' > 0)$

Concave down: $(-1, 1)$

Inflection points: $x = -1$ and $x = 1$.

9. Calculate the integral $\int \tan^3 x \cdot \sec^2 x \, dx$

Hint: $(\tan x)' = \sec^2 x$. u-Sub: $u = \tan x$, $du = \sec^2 x \cdot dx$

$$\int \underbrace{\tan^3 x}_{u} \cdot \underbrace{\sec^2 x \, dx}_{du} = \int u^3 \, du = \frac{1}{4} u^4 + C$$

$$= \boxed{\frac{1}{4} \cdot (\tan x)^4 + C}$$

u back to x.

10. Calculate the integral $\int_0^{\pi/4} \tan x \cdot \sec x + 2x \, dx$

$$\int_0^{\frac{\pi}{4}} \tan x \cdot \sec x + 2x \cdot dx$$

$$= \sec x + x^2 \Big|_0^{\frac{\pi}{4}}$$

$$= \sec\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 - \left(\sec 0 + 0^2\right)$$

$$= \boxed{\frac{2}{\sqrt{2}} + \left(\frac{\pi}{4}\right)^2 - 1}$$

$\sec x = \frac{1}{\cos x}$.

$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$.

$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

11. Find the area of the region enclosed by the graphs of the equations $y = x + 4$ and $y = x^2 - x + 1$.

Intersections: $y = x + 4 = x^2 - x + 1 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = -1,\ x = 3$.

Interval: $[-1, 3]$. Plug in $x = 0$. $\underbrace{y = x + 4}_{\text{Top}} = 0 + 4 > \underbrace{y = x^2 - x + 1}_{\text{Bot}} = 0^2 - 0 + 1$

$$\text{Area} = \int_{-1}^{3} \text{Top} - \text{Bot} \, dx$$

$$= \int_{-1}^{3} (x+4) - (x^2 - x + 1) \, dx$$

$$= \int_{-1}^{3} 2x + 3 - x^2 \, dx = x^2 + 3x - \frac{1}{3}x^3 \Big|_{-1}^{3} = 3^2 + 3\cdot 3 - \frac{1}{3}3^3 - \left((-1)^2 + 3(-1)\right) - \frac{1}{3}(-1)^3)$$

$$= 9 + 9 - 9 - \left(1 - 3 + \frac{1}{3}\right)$$

$$= 9 + 2 - \frac{1}{3} = \boxed{\frac{32}{3}}$$