

A Provably Accurate Algorithm for Recovering Compactly Supported Smooth Functions from Spectrogram Measurements

Michael Perlmutter^{1*}, Nada Sissouno^{2,3}, Aditya Viswantathan⁴, Mark Iwen⁵

¹Michigan State University, Dept. of CMSE

²Technical University of Munich, Dept. of Mathematics

³Helmholtz Center Munich, Mathematical Imaging and Data Analysis, ICT

⁴University of Michigan-Dearborn, Dept. of Mathematics and Statistics

⁵Michigan State University, Dept. of Mathematics and Dept. of CMSE.

*Corresponding Author perlm6@msu.edu.

Abstract—We present an algorithm which is closely related to direct phase retrieval methods that have been shown to work well empirically [1], [2] and prove that it is guaranteed to recover (up to a global phase) a large class of compactly supported smooth functions from their spectrogram measurements. As a result, we take a first step toward developing a new class of practical phaseless imaging algorithms capable of producing provably accurate images of a given sample after it is masked by just a few shifts of a fixed periodic grating.

Index Terms—phase retrieval, phaseless imaging, spectrogram inversion, coded diffraction patterns, Short Time Fourier Transform (STFT) magnitude measurements.

I. INTRODUCTION

Motivated by the plethora of phaseless imaging applications that involve the inversion of spectrogram measurements (see, e.g., [3]), we consider the recovery of a smooth function $f : \mathbb{R} \rightarrow \mathbb{C}$ with support contained in $[-\pi, \pi]$ from a finite set of continuous spectrogram measurements of the form

$$Y_{\omega, \ell} := \left| \int_{-\infty}^{\infty} f(x) m \left(x - \frac{2\pi}{L} \ell \right) e^{-ix\omega} dx \right|^2. \quad (1)$$

Here m is a known trigonometric polynomial, and we use d integer frequencies ω and L shifts $\frac{2\pi}{L} \ell$. In this paper, we present an algorithm that will reconstruct f , up to a global phase multiple, by approximating the d lowest frequency Fourier series coefficients of f restricted to $[-\pi, \pi]$.

A. Notation

Let $k \geq 4$. Let d be odd and δ be even with $4\delta \leq d$. Let $\rho \leq \delta$ be even, L divide d , and $L = \rho + \kappa$ for some $2 \leq \kappa \leq \rho$. For n odd, let $[n]_c := \left[\frac{1-n}{2}, \frac{n-1}{2} \right] \cap \mathbb{Z}$ be the set of n consecutive integers centered at the origin, and let

$$\Omega := [d]_c, \quad \mathcal{B} := [d - \delta]_c \quad \text{and} \quad \mathcal{L} := [L]_c.$$

For vectors \mathbf{x} and \mathbf{y} , we let $\mathbf{x} \circ \mathbf{y}$ and $\frac{\mathbf{x}}{\mathbf{y}}$ be their componentwise product and quotient, and for $\ell \in \mathbb{Z}$, we let S_ℓ be the circular shift operator defined by $(S_\ell \mathbf{x})_p = x_{p+\ell}$ for $\mathbf{x} = (x_p)_{p \in \Omega}$ (where the addition $p + \ell$ is interpreted to mean the unique element of Ω which is equivalent to $p + \ell$

modulo d). We let \mathbf{F}_d be the $d \times d$ Fourier matrix with entries $(\mathbf{F}_d)_{i,j} = e^{-\frac{2\pi i i j}{d}}$ for $i, j \in \Omega$, and similarly let \mathbf{F}_L be the $L \times L$ Fourier matrix with indexes in \mathcal{L} . We will use C to denote an arbitrary constant which depends only on f and m (and in particular does not depend on d).

B. Main Result

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a C^k -smooth function, $k \geq 4$, with

$$\text{supp}(f) \subseteq [-\pi, \pi].$$

For $x \in [-\pi, \pi]$, we will write $f(x)$ as its Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx},$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$. We will let

$$D_n := \max_{|n' - n| < \kappa/2} |\hat{f}(n')|,$$

and assume that $D_n \geq D_{n'}$ whenever $|n| \leq |n'|$.

Remark 1. Under this assumption, for all $|a| < |n|$, there exists n' such that $|a - n'| < \kappa/2$ and $|\hat{f}(n')| \geq |\hat{f}(n)|$.

Let $m(x)$ be a trigonometric polynomial of the form

$$m(x) = \sum_{p=-\rho/2}^{\rho/2} \hat{m}(p) e^{ipx}, \quad (2)$$

and let $\mathbf{Y} = (Y_{\omega, \ell})_{\omega \in \Omega, \ell \in \mathcal{L}}$ be a $d \times L$ matrix of measurements with entries defined as in (1). The central focus of this paper is Algorithm 1 which allows one to reconstruct the signal $f(x)$ from \mathbf{Y} along with the following theorem guaranteeing its convergence as $d \rightarrow \infty$.

Theorem 1. Let μ be the mask dependent constant defined below in (6). If $\mu > 0$, then the output of Algorithm 1, $f_e(x)$, satisfies

$$\begin{aligned} & \min_{\theta \in [0, 2\pi]} \left\| e^{i\theta} f(x) - f_e(x) \right\|_{L^2(-\pi, \pi)} \\ & \leq C \left(\frac{\rho^{1/2} \delta^{1/4}}{\mu^{1/2}} \left(\frac{1}{d} \right)^{(k-3)/2} + \left(\frac{1}{d} \right)^{(k-2)/2} \right). \end{aligned}$$

Remark 2. By imitating the arguments of [8], Proposition 4.1, one may check that it is relatively simple to construct masks such that μ is strictly positive.

C. Related Work

To the best of our knowledge, Algorithm 1 presented here¹ is the first numerical method theoretically guaranteed to accurately recover a complex-valued function f as above up to a constant phase multiple from STFT magnitude measurements of the form (1). Perhaps the most closely related result to ours is that of Thakur [4] who gives an algorithm for the reconstruction of real-valued bandlimited functions up to a global sign. Gröchenig [5] also considers/surveys similar results in shift-invariant spaces. Other related work includes that of Alaifari et al. [6] which proves (among other things) that one can not hope to stably recover a periodic function up to a single global phase using a trigonometric polynomial mask of degree $\rho/2$ as done below unless its Fourier series coefficients do not vanish on any ρ consecutive integer frequencies in between two other frequencies with nonzero coefficients. This helps motivate the quantity D_n as well as the assumption that $D_n \geq D_{n'}$ whenever $|n| \leq |n'|$. See [7] for similar considerations in the discrete setting.

II. DISCRETIZATION

Let $P_{\mathcal{B}}f$ be the partial Fourier series

$$P_{\mathcal{B}}f(x) := \sum_{n \in \mathcal{B}} \widehat{f}(n) e^{inx},$$

and let $\mathbf{T} := (T_{\omega, \ell})_{\omega \in \Omega, \ell \in \mathcal{L}}$ denote the matrix of measurements obtained by replacing f with $P_{\mathcal{B}}f$ in (1), i.e.,

$$T_{\omega, \ell} := \left| \int_{-\pi}^{\pi} P_{\mathcal{B}}f(x) m\left(x - \frac{2\pi}{L}\ell\right) e^{-ix\omega} dx \right|^2. \quad (3)$$

Our method is based on showing that \mathbf{Y} is well-approximated by \mathbf{T} and by representing $P_{\mathcal{B}}f(x)$ and $m(x)$ with vectors $\mathbf{x} = (x_p)_{p \in \Omega}$ and $\mathbf{y} = (y_p)_{p \in \Omega}$ defined by

$$x_p := P_{\mathcal{B}}f\left(\frac{2\pi p}{d}\right) \quad \text{and} \quad y_p := m\left(\frac{2\pi p}{d}\right).$$

We will also define $\mathbf{u} = (u_p)_{p \in \Omega}$ and $\mathbf{v} = (v_p)_{p \in \Omega}$ by

$$u_p := \widehat{f}(p) \mathbb{1}_{p \in \mathcal{B}} \quad \text{and} \quad v_p := \widehat{m}(p) \mathbb{1}_{|p| \leq \rho/2}, \quad (4)$$

where $\mathbb{1}_{p \in \mathcal{B}}$ and $\mathbb{1}_{|p| \leq \rho/2}$ are standard indicator functions. We note that the Fourier transforms of \mathbf{x} and \mathbf{y} satisfy

$$\mathbf{F}_d \mathbf{x} =: \widehat{\mathbf{x}} = d\mathbf{u} \quad \text{and} \quad \mathbf{F}_d \mathbf{y} =: \widehat{\mathbf{y}} = d\mathbf{v}. \quad (5)$$

Let μ be a mask-dependent constant defined by

$$\mu := \inf_{d \geq \rho} \min_{|p| < \kappa, q \in \Omega} |\mathbf{F}_d(\mathbf{v} \circ S_p \bar{\mathbf{v}})_q| =: \inf_{d \in \mathbb{N}} \mu_d. \quad (6)$$

¹Numerical results available at <https://bitbucket.org/charms/blockpr>

We note that in light of (5) we have

$$\nu_d := \min_{|p| < \kappa, q \in \Omega} \left| \mathbf{F}_d(\widehat{\mathbf{y}} \circ S_p \bar{\widehat{\mathbf{y}}})_q \right| = d^2 \mu_d \geq d^2 \mu. \quad (7)$$

The following lemma shows that the integral in (3) can be replaced by a discrete sum. It is proved by expanding $P_{\mathcal{B}}f$ and m as trigonometric polynomials and using the fact that

$$2\pi \sum_{p \in \Omega} e^{2\pi i p j / d} = d \int_{-\pi}^{\pi} e^{ijx} dx \quad \forall j \in \Omega.$$

Lemma 1. Let $\ell \in \mathcal{L}$, $\omega \in \Omega$, and let $\tilde{\ell} = \frac{2\pi \ell}{L}$. Then,

$$\int_{-\pi}^{\pi} P_{\mathcal{B}}f(x) m\left(x - \tilde{\ell}\right) e^{-ix\omega} dx = \frac{2\pi}{d} \sum_{p \in \Omega} x_p y_{p - \ell \frac{d}{L}} e^{-2\pi i \omega p / d}.$$

We may use Lemma 1 to prove the following result which shows that \mathbf{T} converges to \mathbf{Y} as $d \rightarrow \infty$.

Lemma 2. Let $\mathbf{E} := \mathbf{Y} - \mathbf{T}$. Then

$$\|\mathbf{E}\|_{\infty} \leq C \rho \left(\frac{1}{d}\right)^k, \quad \text{and} \quad (8)$$

$$E_{\omega, \ell} = 0 \quad \text{whenever} \quad |\omega| \leq \frac{d-1}{2} - \delta. \quad (9)$$

Proof. For $\omega \in \Omega$ and $\ell \in \mathcal{L}$, let

$$M_{\omega, \ell} := \int_{-\pi}^{\pi} f(x) m\left(x - \frac{2\pi}{d}\ell\right) e^{-ix\omega} dx, \quad \text{and}$$

$$U_{\omega, \ell} := \int_{-\pi}^{\pi} P_{\mathcal{B}}f(x) m\left(x - \frac{2\pi}{d}\ell\right) e^{-ix\omega} dx.$$

It suffices to show that

$$|U_{\omega, \ell}| \leq C \quad (10)$$

$$\text{and} \quad |E'_{\omega, \ell}| \leq C \rho \left(\frac{1}{d}\right)^k. \quad (11)$$

Then, letting $E'_{\omega, \ell} := M_{\omega, \ell} - U_{\omega, \ell}$, we will have

$$\begin{aligned} |E_{\omega, \ell}| &= \left| |M_{\omega, \ell}|^2 - |U_{\omega, \ell}|^2 \right| \\ &= (|M_{\omega, \ell}| + |U_{\omega, \ell}|) \left| |M_{\omega, \ell}| - |U_{\omega, \ell}| \right| \\ &\leq (2|U_{\omega, \ell}| + |E'_{\omega, \ell}|) |E'_{\omega, \ell}| \\ &\leq C \left(1 + \rho \left(\frac{1}{d}\right)^k \right) \rho \left(\frac{1}{d}\right)^k \\ &\leq C \rho \left(\frac{1}{d}\right)^k. \end{aligned}$$

Since $m(x)$ is a trigonometric polynomial, we see

$$\|\mathbf{y}\|_{\infty} \leq \|m\|_{\infty} \leq C,$$

and since f is \mathcal{C}^2 -smooth and compactly supported, we have

$$\|\mathbf{x}\|_{\infty} \leq \|P_{\mathcal{B}}f\|_{\infty} \leq \sum_{n \in \mathbb{Z}} |\widehat{f}(n)| \leq C.$$

Therefore, using Lemma 1, we see that

$$|U_{\omega, \ell}| = \left| \frac{2\pi}{d} \sum_{p \in \Omega} x_p y_{p - \ell \frac{d}{L}} e^{-2\pi i \omega p / d} \right| \leq C, \quad (12)$$

and so (10) follows. To prove (11), we note that

$$f(x) - P_{\mathcal{B}}f(x) = \sum_{n \notin \mathcal{B}} \widehat{f}(n) e^{in x},$$

and therefore

$$\begin{aligned} E'_{\omega, l} &= \int_{-\pi}^{\pi} (f(x) - P_{\mathcal{B}}f(x)) m \left(x - \frac{2\pi\ell}{L} \right) e^{-i\omega x} dx \\ &= \sum_{n \notin \mathcal{B}} \sum_{p=-\rho/2}^{\rho/2} \widehat{f}(n) \widehat{m}(p) e^{-\frac{2\pi i p \ell}{L}} \int_{-\pi}^{\pi} e^{i(n+p-\omega)x} dx. \end{aligned}$$

The inner integral is zero unless $\omega = n + p$. Therefore,

$$\begin{aligned} |E'_{\omega, l}| &\leq 2\pi \sum_{n \notin \mathcal{B}, |\omega-n| \leq \rho/2} |\widehat{f}(n)| |\widehat{m}(\omega-n)| \\ &\leq 2\pi \sup_{n \notin \mathcal{B}} \{ |\widehat{f}(n)| \} \rho \sup_{|n| \leq \rho/2} |\widehat{m}(n)| \\ &\leq C\rho \left(\frac{1}{d} \right)^k, \end{aligned}$$

where we used the facts $\widehat{f}(n) = \mathcal{O}(n^{-k})$ and that $n > \frac{d}{4}$ for all $n \notin \mathcal{B}$. This proves (8). Equation (9) follows from noting that the condition $\omega = n + p$ can never hold when $n \notin \mathcal{B}$, $|\omega| \leq \frac{d-1}{2} - \delta$ and $|p| \leq \rho/2$. \square

III. WIGNER DECONVOLUTION

In this section, we apply a discrete, aliased Wigner deconvolution approach, similar to Section 3 of [8], to solve for a portion of the Fourier autocorrelation matrix $\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*$. It follows from (12) that

$$T_{\omega, \ell} = \frac{4\pi^2}{d^2} \left| \sum_{p \in \Omega} x_p y_{p-\ell \frac{d}{L}} e^{-2\pi i \omega p/d} \right|.$$

Up to a scaling factor of $4\pi^2/d^2$, these measurements coincide with the measurements considered in [8].

Let $\widetilde{\mathbf{T}} := \mathbf{F}_L \mathbf{T}^T \mathbf{F}_d^T$, and let $\widetilde{\mathbf{E}} := \mathbf{F}_L \mathbf{E}^T \mathbf{F}_d^T$. Since $\frac{1}{\sqrt{d}} \mathbf{F}_d$ and $\frac{1}{\sqrt{L}} \mathbf{F}_L$ are unitary, we may use Lemma 2 to see

$$\|\widetilde{\mathbf{E}}\|_F \leq \sqrt{dL} \|\mathbf{E}\|_F \leq \sqrt{2d\delta L} \|\mathbf{E}\|_{\infty} \leq CL\rho\delta^{1/2} \left(\frac{1}{d} \right)^{k-1/2}. \quad (13)$$

It follows from Theorem 4, Equation 3.2, of [8] that

$$\begin{aligned} \widetilde{T}_{\ell, \omega} - \widetilde{E}_{\ell, \omega} &= \\ \frac{4\pi^2 L}{d^4} \sum_{p \in [\frac{d}{L}]_c} \left(\mathbf{F}_d \left(\widehat{\mathbf{x}} \circ S_{pL-\ell} \widehat{\mathbf{x}} \right) \right)_{\omega} \left(\mathbf{F}_d \left(\widehat{\mathbf{y}} \circ S_{\ell-pL} \widehat{\mathbf{y}} \right) \right)_{\omega}, \end{aligned}$$

where, as in Section I, $(S_{\ell} \mathbf{x})_p = \mathbf{x}_{p+\ell}$ for all $\ell \in \mathbb{Z}$. By construction, we have that $\text{supp}(\widehat{\mathbf{y}}) \subseteq [\rho+1]_c$. Therefore, if $1 - \kappa \leq \ell \leq \kappa - 1$, we may use the same reasoning as in the proof of Lemma 10 of [8], to see that $\widehat{\mathbf{y}} \circ S_{\ell-pL} \widehat{\mathbf{y}} = 0$ except for when $p = 0$. Therefore,

$$\widetilde{T}_{\ell, \omega} - \widetilde{E}_{\ell, \omega} = \frac{4\pi^2 L}{d^4} \left(\mathbf{F}_d \left(\widehat{\mathbf{x}} \circ S_{-\ell} \widehat{\mathbf{x}} \right) \right)_{\omega} \left(\mathbf{F}_d \left(\widehat{\mathbf{y}} \circ S_{\ell} \widehat{\mathbf{y}} \right) \right)_{\omega}.$$

Changing variables $\ell \rightarrow -\ell$ we see that

$$\left(\mathbf{F}_d \left(\widehat{\mathbf{x}} \circ S_{\ell} \widehat{\mathbf{x}} \right) \right)_{\omega} = \frac{d^4}{4\pi^2 L} \left(\frac{\widetilde{T}_{-\ell, \omega} - \widetilde{E}_{-\ell, \omega}}{\left(\mathbf{F}_d \left(\widehat{\mathbf{y}} \circ S_{-\ell} \widehat{\mathbf{y}} \right) \right)_{\omega}} \right),$$

and so

$$\widehat{\mathbf{x}} \circ S_{\ell} \widehat{\mathbf{x}} = \frac{d^4}{4\pi^2 L} \mathbf{F}_d^{-1} \left(\frac{\widetilde{\mathbf{T}}_{-\ell} - \widetilde{\mathbf{E}}_{-\ell}}{\left(\mathbf{F}_d \left(\widehat{\mathbf{y}} \circ S_{-\ell} \widehat{\mathbf{y}} \right) \right)} \right), \quad (14)$$

where \mathbf{M}_j denotes the j -th column of a matrix \mathbf{M} and we define vector division componentwise.

Let $T_{\kappa} : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d \times d}$ be the restriction operator

$$T_{\kappa}(\mathbf{M})_{ij} = \begin{cases} M_{i,j} & \text{if } |i-j| < \kappa \\ 0 & \text{otherwise} \end{cases}.$$

We may rewrite (14) in matrix form as

$$T_{\kappa}(\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*) = \mathbf{X} + \mathbf{N}, \quad (15)$$

where the matrix $\mathbf{X} = (X_{i,j})_{i,j \in \Omega}$ has entries defined by

$$X_{i,j} = \begin{cases} \frac{d^4}{4\pi^2 L} \left(\mathbf{F}_d^{-1} \left(\frac{\widetilde{\mathbf{T}}_{i-j}}{\left(\mathbf{F}_d \left(\widehat{\mathbf{y}} \circ S_{i-j} \widehat{\mathbf{y}} \right) \right)} \right) \right)_i & \text{if } |i-j| < \kappa \\ 0 & \text{otherwise} \end{cases}, \quad (16)$$

and \mathbf{N} is defined similarly using $\widetilde{\mathbf{E}}$ in place of $\widetilde{\mathbf{T}}$. Let $\mathbf{R} = (R_{i,j})_{i \in \Omega, j \in [2\kappa-1]_c}$ be the $d \times (2\kappa-1)$ matrix with entries $R_{i,j} = N_{i,i+j}$ so that the columns of \mathbf{R} are the diagonal bands of \mathbf{N} within κ of the main diagonal. By (7), we may bound the ℓ^2 -norm of each column of \mathbf{R} by

$$\begin{aligned} \|\mathbf{R}_j\|_2 &= \left\| \frac{d^4}{4\pi^2 L} \mathbf{F}_d^{-1} \left(\frac{\widetilde{\mathbf{E}}_{-j}}{\left(\mathbf{F}_d \left(\widehat{\mathbf{y}} \circ S_{-j} \widehat{\mathbf{y}} \right) \right)} \right) \right\|_2 \\ &\leq \frac{d^{7/2}}{4\pi^2 L \nu_d} \|\widetilde{\mathbf{E}}_{-j}\|_2 \leq \frac{d^{3/2}}{4\pi^2 L \mu} \|\widetilde{\mathbf{E}}_{-j}\|_2. \end{aligned}$$

Since \mathbf{N} is a banded matrix, (13) implies

$$\|\mathbf{N}\|_F = \|\mathbf{R}\|_F \leq \frac{d^{3/2}}{4\pi^2 L \mu} \|\widetilde{\mathbf{E}}\|_F \leq C \frac{\rho \delta^{1/2}}{\mu} \left(\frac{1}{d} \right)^{k-2}. \quad (17)$$

Dividing both sides of (15) by d^2 , using (5), and applying the Hermitianizing operator $H(\mathbf{M}) = \frac{1}{2}(\mathbf{M} + \mathbf{M}^*)$ yields

$$T_{\kappa}(\mathbf{u}\mathbf{u}^*) = \mathbf{A} + \widetilde{\mathbf{N}}, \quad (18)$$

where $\mathbf{A} = d^{-2}H(\mathbf{X})$, and $\widetilde{\mathbf{N}} := d^{-2}H(\mathbf{N})$. By (17), and the triangle inequality, we have

$$\|\widetilde{\mathbf{N}}\|_F \leq C \frac{\rho \delta^{1/2}}{\mu} \left(\frac{1}{d} \right)^k. \quad (19)$$

IV. ANGULAR SYNCHRONIZATION

In this section, we will use a greedy angular synchronization approach to recover the Fourier coefficients of f . For each $n \in \mathcal{B}$, the greedy algorithm, Algorithm 2, outputs a

sequence $\{n_\ell\}_{\ell=0}^b$ where $n_0 = \arg \max_{n \in \mathcal{B}} a_n$ and $n_b = n$. Given that sequence, we let

$$\alpha_n := \sum_{l=0}^{b-1} \arg(A_{n_{\ell+1}, n_\ell}).$$

To understand this definition, let

$$\theta_0 := \arg(\widehat{f}(n_0)) \quad \text{and} \quad \tau_n := \sum_{l=0}^{b-1} \arg((\mathbf{u}\mathbf{u}^*)_{n_{\ell+1}, n_\ell}).$$

Then, we have $\tau_n = \arg(\widehat{f}(n)) - \theta_0$, and therefore

$$e^{-i\theta_0} \widehat{f}(n) = |\widehat{f}(n)| e^{i\tau_n}$$

for all $n \in \mathcal{B}$. (Note that n_0 does not depend on n .) Since \mathbf{A} is a noisy approximation of $\mathbf{u}\mathbf{u}^*$, we intuitively view α_n as a noisy approximation of τ_n (up to a phase shift θ_0). Lemma 3 will show that this intuition is correct when $|\widehat{f}(n)|$ is sufficiently large. Due to [9, Lemma 3], for all $n \in \mathcal{B}$ we have

$$\left| \sqrt{|A_{n,n}|} - |\widehat{f}(n)| \right|^2 \leq 3 \|\widetilde{\mathbf{N}}\|_\infty. \quad (20)$$

Therefore, we set $a_n := \sqrt{|A_{n,n}|}$ and define the output of Algorithm 1 to be the trigonometric polynomial

$$f_e(x) := \sum_{n \in \mathcal{B}} a_n e^{i\alpha_n} e^{inx}.$$

The following lemma shows that α_n is indeed a good approximation of τ_n when $|\widehat{f}(n)|$ is sufficiently large. Its proof is nearly identical to the proof of [9, Lemma 4], but uses Lemma 4 stated below in place of the ‘‘flat vector’’ condition considered there.

Lemma 3. *Let L_f be the set*

$$L_f = \{n \in \mathcal{B} : |\widehat{f}(n)|^2 \geq 48 \|\widetilde{\mathbf{N}}\|_\infty\}.$$

Then, for all $n \in L_f$

$$|e^{i\tau_n} - e^{i\alpha_n}| \leq \frac{2\pi d \|\widetilde{\mathbf{N}}\|_\infty}{|\widehat{f}(n)|^2}.$$

As mentioned above, the key to modifying the proof of [9, Lemma 4] in order to prove Lemma 3 is the following lemma, which shows that Algorithm 2 will only select entries n_ℓ corresponding to large Fourier coefficients.

Lemma 4. *Let $n \in L_f$, and let $\{n_\ell\}_{\ell=0}^b$ be the sequence output by Algorithm 2. Then,*

$$|\widehat{f}(n_\ell)| \geq \frac{|\widehat{f}(n)|}{2} \quad \text{for all } 0 \leq \ell \leq b.$$

Proof. When $\ell = b$, the claim is immediate. For $0 \leq \ell \leq b-1$, we have $a_{n_\ell} = \max_{m \in I_\ell} a_m$ for some interval I_ℓ of length 2κ , which is centered at some $|a| \leq |n|$. Therefore, letting $\epsilon = \sqrt{3 \|\widetilde{\mathbf{N}}\|_\infty}$, we see that by (20) and Remark 1,

$$|\widehat{f}(n_\ell)| \geq \max_{m \in I_\ell} a_m - \epsilon \geq \max_{m \in I_\ell} |\widehat{f}(m)| - 2\epsilon \geq |\widehat{f}(n)| - 2\epsilon.$$

The result follows by noting that $\epsilon < \frac{|\widehat{f}(n)|}{4}$ for $n \in L_f$. \square

Together, (20) and Lemma 3 allow us to prove the following lemma showing that $f_e(x)$ approximates $P_{\mathcal{B}}f(x)$.

Lemma 5. *The output of Algorithm 1 satisfies*

$$\left\| e^{-i\theta_0} P_{\mathcal{B}}f(x) - f_e(x) \right\|_{L^2(-\pi, \pi)} \leq C d^{\frac{3}{2}} \sqrt{\|\widetilde{\mathbf{N}}\|_\infty}.$$

Proof. Recall the vector \mathbf{u} defined in (4) and, for $n \in \Omega$, let

$$u'_n := a_n e^{i\alpha_n} \quad \text{and} \quad u''_n := |u_n| e^{i\alpha_n}.$$

By construction, for all $n \notin \mathcal{B}$ we have $a_n = u_n = 0$. Therefore the supports of $\mathbf{u}' := (u'_n)_{n \in \Omega}$ and $\mathbf{u}'' := (u''_n)_{n \in \Omega}$ are contained in \mathcal{B} . By Parseval’s identity, we see

$$\begin{aligned} & \left\| e^{-i\theta_0} P_{\mathcal{B}}f(x) - \sum_{n \in \mathcal{B}} a_n e^{i\alpha_n} e^{inx} \right\|_{L^2(-\pi, \pi)} \\ &= \left\| e^{-i\theta_0} \sum_{n \in \mathcal{B}} u_n e^{inx} - \sum_{n \in \mathcal{B}} u'_n e^{inx} \right\|_{L^2(-\pi, \pi)} \\ &\leq \sqrt{2\pi} \|e^{-i\theta_0} \mathbf{u} - \mathbf{u}'\|_{\ell_2} \\ &\leq \sqrt{2\pi} \|e^{-i\theta_0} \mathbf{u} - \mathbf{u}''\|_{\ell_2} + \sqrt{2\pi} \|\mathbf{u}'' - \mathbf{u}'\|_{\ell_2} \\ &=: I_1 + I_2. \end{aligned}$$

Using Lemma 3 and the fact that $|e^{i\tau_n} - e^{i\alpha_n}| \leq 2$, we have

$$\begin{aligned} I_1^2 &= 2\pi \sum_{n \in \mathcal{B}} |u_n|^2 |e^{i\tau_n} - e^{i\alpha_n}|^2 \\ &\leq C \sum_{n \in \mathcal{B} \setminus L_f} |u_n|^2 + C \sum_{n \in L_f} d^2 \|\widetilde{\mathbf{N}}\|_\infty^2 |\widehat{f}(n)|^{-2} \\ &\leq C d \|\widetilde{\mathbf{N}}\|_\infty + C \sum_{n \in L_f} d^2 \|\widetilde{\mathbf{N}}\|_\infty \\ &\leq C d^3 \|\widetilde{\mathbf{N}}\|_\infty. \end{aligned}$$

To estimate I_2 , we recall (20) and note

$$I_2^2 = 2\pi \sum_{n \in \mathcal{B}} \left| |u_n| - a_n \right|^2 \leq C d \|\widetilde{\mathbf{N}}\|_\infty. \quad \square$$

We now use Lemma 5 as well as the uniform convergence of the partial Fourier series $P_{\mathcal{B}}f$ to prove Theorem 1.

Proof. [The Proof of Theorem 1] By the triangle inequality,

$$\begin{aligned} & \min_{\theta \in [0, 2\pi]} \left\| e^{i\theta} f(x) - \sum_{n \in \mathcal{B}} a_n e^{i\alpha_n} e^{inx} \right\|_{L^2(-\pi, \pi)} \\ &\leq \|f - P_{\mathcal{B}}f\|_{L^2(-\pi, \pi)} \\ &\quad + \left\| e^{-i\theta_0} P_{\mathcal{B}}f(x) - \sum_{n \in \mathcal{B}} a_n e^{i\alpha_n} e^{inx} \right\|_{L^2(-\pi, \pi)}, \end{aligned}$$

where $\theta_0 = \arg(\widehat{f}(n_0))$. By (19) and Lemma 5, we have

$$\left\| e^{-i\theta_0} P_{\mathcal{B}}f(x) - f_e(x) \right\|_2 \leq C \frac{\rho^{1/2} \delta^{1/4}}{\mu^{1/2}} \left(\frac{1}{d} \right)^{(k-3)/2}.$$

To bound the first term we see that by Parseval's identity, the fact that $d > 4\delta$, and the fact that $|\widehat{f}(n)| = \mathcal{O}(d^{-k})$

$$\|f - P_{\mathcal{B}}f\|_2^2 = 2\pi \sum_{n \notin \mathcal{B}} |\widehat{f}(n)|^2 \leq C \sum_{n \geq \frac{d}{4}} \frac{1}{n^{2k}} \leq C \frac{1}{d^{2k-1}}. \square$$

Algorithm 1 Wigner Deconvolution and Angular Synchronization for Bandlimited Masks

Inputs

- 1) Matrix $\mathbf{Y} = (Y_{\omega, \ell})_{\omega \in \Omega, \ell \in \mathcal{L}}$ of spectrogram measurements defined as in (1).
- 2) Trigonometric polynomial mask of the form (2).

Steps

- 1) Define vector $\mathbf{y} = (y_p)_{p \in \Omega}$ by $y_p = m\left(\frac{2\pi p}{d}\right)$.
- 2) Let $\kappa = L - \rho$, and for $1 - \kappa \leq \ell \leq \kappa - 1$ estimate

$$\mathbf{F}_d \left(\widehat{\mathbf{x}} \circ S_{\ell} \widehat{\widehat{\mathbf{x}}} \right) \approx \frac{d^4}{4\pi^2 L} \left(\frac{(\mathbf{F}_L \mathbf{Y}^T \mathbf{F}_d^T)_{-\ell}}{(\mathbf{F}_d(\widehat{\mathbf{y}} \circ S_{-\ell} \widehat{\widehat{\mathbf{y}}}))} \right).$$

- 3) Invert the Fourier transforms above to recover estimates of the vectors $\widehat{\mathbf{x}} \circ S_{\ell} \widehat{\widehat{\mathbf{x}}}$.
- 4) Organize these vectors into a banded matrix, \mathbf{X} described as in (16).
- 5) Hermitianize \mathbf{X} and divide by d^2 to obtain the matrix $\mathbf{A} = (A_{i,j})_{i,j \in \Omega}$ as described in (18).
- 6) Estimate $|\widehat{f}(n)| \approx a_n = \sqrt{|A_{n,n}|}$.
- 7) For $n \in \mathcal{B}$, choose $\{n_{\ell}\}_{\ell=0}^b$ according to Algorithm 2.
- 8) Approximate

$$\arg \left(\widehat{f}(n) \right) \approx \alpha_n = \sum_{\ell=0}^{b-1} \arg \left(A_{n_{\ell+1}, n_{\ell}} \right).$$

Output

An approximation of f given by

$$f_e(x) = \sum_{n \in \mathcal{B}} a_n e^{i\alpha_n} e^{inx}.$$

V. FUTURE WORK

The work here shows that, under suitable regularity assumptions, we may recover a continuous signal $f(x)$ from a $d \times L$ matrix of phaseless measurements. We believe that this paper naturally opens up several research directions for future work. Firstly, one might replace the assumption that $m(x)$ is a trigonometric polynomial with the assumption that $m(x)$ is compactly supported in space. This would lead to a measurement setup closely related to ptychographic imaging. Also, in [8], it is shown that in the discrete setting, a discrete Wigner deconvolution approach can be applied to a $K \times L$ measurement matrix for some $K < d$ and that this approach is robust to additive noise. It is likely

Algorithm 2 Greedy Entry Selection

Inputs

- 1) Vector of amplitudes $\mathbf{a} = (a_n)_{n \in \Omega}$, $a_n = \sqrt{|A_{n,n}|}$.
- 2) Entry $n \in \mathcal{B}$.

Steps

- 1) Choose $n_0 = \arg \max_{n \in \mathcal{B}} a_n$.
- 2) Let $b = 0$.
- 3) While: $|n - n_b| > \kappa/2$
- 4) If: $n > n_b$, let $n_{b+1} \leftarrow \arg \max_{n_b < m < n_b + \kappa} a_m$
- 5) If: $n < n_b$, let $n_{b+1} \leftarrow \arg \max_{n_b - \kappa < m < n_b} a_m$
- 6) $b \leftarrow b + 1$
- 7) $n_b \leftarrow n$

Output

A sequence $\{n_{\ell}\}_{\ell=0}^b$, $|n_{\ell+1} - n_{\ell}| < \kappa/2$, $n_b = n$, $b \leq d$.

that analogous techniques can be applied in the continuous setting when the matrix \mathbf{Y} is subsampled in frequency and corrupted by additive noise. Lastly, one might also extend these results to functions of two variables $f(x, y)$.

ACKNOWLEDGMENTS

Mark Iwen was supported in part by NSF DMS 1912706. Nada Sissouno was supported in part by the Entrepreneurial Awards of the Global Challenges for Women program in Math Science.

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