**Standard Response Questions**. Show all work to receive credit. Please **BOX** your final answer.

1. Calculate the following limits or show that they do not exist:

(a) (4 points) $\lim_{x\to -1} \frac{x^2-1}{x+1} =$

**Solution:**

$$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} = \lim_{x\to -1} x-1 = -2$$

(b) (5 points) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{|x|}\right) =$

**Solution:** For $x > 0$, that is $x \to 0^+$ we have

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0^+} (0) = 0$$

For $x < 0$, that is $x \to 0^-$ though we have

$$\lim_{x\to 0^-} \left(\frac{1}{x} + \frac{1}{x}\right) = \lim_{x\to 0^-} \left(\frac{2}{x}\right) = -\infty$$

Since these are not equal the two sided limit **does not exist**.

2. (5 points) Find the value of $a$ that makes the function continuous at $x = 0$:

$$f(x) = \begin{cases} \frac{\sin(-8x)}{x} & \text{if } x < 0 \\ 3x + 6a - 7 & \text{if } x \geq 0 \end{cases}$$

**Solution:**

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(-8x)}{x} = \lim_{x\to 0^-} \frac{\sin(-8x)}{-8x} \cdot (-8) = -8$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x + 6a - 7 = 6a - 7$$

For $f$ to be continuous at 0 the two sided limit must exist and so

$$\begin{aligned} -8 &= 6a - 7 \\ -1 &= 6a \\ -1/6 &= a \end{aligned}$$

3. (7 points) The length of a rectangle is decreasing at a rate of 4 cm/s and its width is increasing at a rate of 5 cm/s. When the length is 12 cm and the width 10 cm, how fast is the area of the rectangle changing? Is the area increasing or decreasing at that time? (*Include Units*)

**Solution:**

$$A(t) = l(t)w(t)$$
$$A'(t) = l'(t)w(t) + l(t)w'(t)$$

now plugging in our specific values we have

$$A'(t) = (-4)(10) + (12)(5) = 20 \text{ cm}^2/\text{s}$$

Since $20 > 0$ this indicates the area is increasing at that time.

4. Given $f(x) = x^2 + 10\sin x$:

(a) (1 point) Indicate the interval where the function is continuous.

**Solution:** $(-\infty, \infty)$

(b) (4 points) Prove that there is a number $c$ such that $f(c) = 1$,
using the Intermediate Value Theorem.

**Solution:** One possible solution is

$$f(0) = 0 < 1$$
$$f(\pi) = \pi^2 > 1$$

So since $f$ is continuous by the IVT there is a $c$ in which $f(c) = 1$.

(c) (2 points) Using (b) state an interval where $c$ can be found.

**Solution:** $c \in (0, \pi)$

5. Compute the derivatives of the following functions: (**DO NOT SIMPLIFY**)

(a) (4 points) $f(x) = x\sec(x)$

**Solution:**

$$f'(x) = \sec(x) + x\sec(x)\tan(x)$$

(b) (4 points) $g(x) = \dfrac{x^3+1}{6x^2+7}$

**Solution:**

$$g'(x) = \frac{(3x^2)(6x^2+7) - (x^3+1)(12x)}{(6x^2+7)^2}$$

6. (6 points) Find the equation of the tangent line to the curve $y = \sin\left(\dfrac{\pi x^2}{4}\right)$ at the point $\left(1, \frac{\sqrt{2}}{2}\right)$.

**Solution:**

$$y'(x) = \cos\left(\frac{\pi x^2}{4}\right) \cdot \left(\frac{\pi x}{2}\right)$$
$$y'(1) = \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\pi}{2}\right) = \frac{\pi\sqrt{2}}{4}$$

So

$$y - \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}(x-1)$$

7. (7 points) Given $y = \sqrt{x}$, use the limit definition of the derivative to compute $y'$.

**Solution:**

$$\begin{aligned} y' &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

8. (7 points) Consider $y^2 + xy + \dfrac{3}{y} = 4 + x^2$. Use implicit differentiation to find $y'$.

**Solution:**

$$\begin{aligned} y^2 + xy + \frac{3}{y} &= 4 + x^2 \\ 2yy' + (y + xy') - \frac{3}{y^2}y' &= 2x \\ 2yy' + xy' - \frac{3}{y^2}y' &= 2x - y \\ y'(2y + x - \frac{3}{y^2}) &= 2x - y \\ y' &= \frac{2x-y}{2y + x - \frac{3}{y^2}} \end{aligned}$$

**Multiple Choice**. Circle the single best answer. No work needed. No partial credit available.

9. (4 points) Given $f(x) = |x - 5|$, which of the following statements is true?

   A. $f(x)$ is continuous and differentiable on $(-\infty, \infty)$.

   B. $f(x)$ is continuous on $(-\infty, \infty)$, and differentiable on $(-\infty, 5) \cup (5, \infty)$.

   C. $f(x)$ is continuous and differentiable on $(-\infty, 5) \cup (5, \infty)$.

   D. $f(x)$ is differentiable on $(-\infty, \infty)$, but not continuous at $x = 5$.

   E. $f(x)$ is not defined at $x = 5$.

10. (4 points) Suppose $f(x)$ is continuous and differentiable and that $f'(x) > 0$ always and $f(0) = 3$. What is true about $f(1)$?

    A. It is possible that $f(1) = 3$

    B. It must be that $f(1) < 3$

    C. It must be that $f(1) > 3$

    D. There is not enough information

11. (4 points) Given $x^2 + y^2 = 9$, which of the following is true?
    ***(Hint**: Use implicit differentiation or sketch the graph.)*

    A. $y' > 0$ always

    B. $y' < 0$ always

    C. $y' > 0$ in the I and III quadrants

    D. $y' > 0$ in the II and IV quadrants

    E. None of the above

12. Suppose the height of an object is modeled by $h(t) = 10t - 2t^2$ m, with time measured in seconds.

(a) (4 points) When does the object reach its maximum height?

A. 2.5 s

B. 25 s

C. 10 s

D. 4 s

E. None of the above.

(b) (4 points) What is the maximum height of the object?

A. 25 m

B. 12.5 m

C. 50 m

D. 2.5 m

E. None of the above.

(c) (4 points) What is the direction of the object at time $t = 4$ s?

A. downward

B. upward

C. There is not enough information.

13. (4 points) Use the squeeze theorem to evaluate: $\lim_{x\to 0}\sqrt{\frac{x^3+x^2}{\pi}}\sin\frac{\pi}{x}$

A. 1

B. 0

C. DNE

D. $\frac{1}{\pi}$

14. (4 points) Calculate the derivative of $f(x) = \cos(\tan x)$

A. $f'(x) = -\sin(\sec^2 x)$

B. $f'(x) = \sin(\tan x)\sec^2 x$

C. $f'(x) = -\sin(\tan x)\sec^2 x$

D. $f'(x) = \cos(\sec^2 x)$

E. $f'(x) = -\sin x \sec^2 x$

15. (4 points) The velocity of a particle moving back and forth along a straight line is given by $v(t) = 2\sin(\pi t) + 3\cos(\pi t)$, where time is measured in seconds.
What does $v'(t) = 2\pi\cos(\pi t) - 3\pi\sin(\pi t)$ represent?

A. The average rate of change of the position of the particle over any 1-second interval

B. The instantaneous rate of change of velocity

C. The speed at which the particle is moving

D. The average rate of change of the velocity of the particle over any 1-second interval

E. The instantaneous rate of change of position

**More Challenging Question(s)**. Show all work to receive credit.

16. Newton's Law of Gravitation says that the magnitude of the force, $F$, exerted by a body of mass $m$ on a body of mass $M$ is

$$F = \frac{GmM}{r^2}$$

where $G$ is the gravitational constant and $r$ is the distance between the bodies.

(a) (4 points) Calculate $\frac{dF}{dr}$.

**Solution:**

$$\frac{dF}{dr} = \frac{-2GmM}{r^3}$$

(b) (2 points) Explain the physical meaning of $\frac{dF}{dr}$.

**Solution:** How force of gravity is changing as the radius changes.

(c) (8 points) Suppose the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How is this force changing when $r = 10,000$ km? (*Include units.*)

**Solution:**

$$\begin{aligned} -2 &= \frac{-2GmM}{20,000^3} \\ 20,000^3 &= GmM \\ \frac{dF}{dr} &= \frac{-2 \cdot 20,000^3}{10,000^3} \\ &= -16 \text{ N/km} \end{aligned}$$

So it is decreasing at a rate of 16 Newtons each kilometer when the $r = 10,000$ km.