The Ninth Annual Herzog Prize Examination

November 7, 1981

Problem 1: Does the inequality

\[(2n)^n + (2n+1)^n \geq (2n+2)^n\]

hold for all positive integers \(n\)?

Problem 2: (L.M. Kelly) A point \(P\) moves on the positive \(x\)-axis and point \(Q\) on the positive \(y\)-axis so that the line \(PQ\) is always tangent to the ellipse

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.\]

Prove that when the triangle \(POQ\) has minimum area the point of tangency will bisect the segment \(PQ\).

(Here \(O\) is the origin.)

Problem 3: (J. Marik) Let \(f, g\) be nondecreasing on \([0,1]\).

Prove that

\[\int_0^1 f(x)\,dx \int_0^1 g(x)\,dx \leq \int_0^1 f(x)g(x)\,dx.\]

Problem 4: (L.M. Kelly) If \(a, b, c\) are real and \(b^2 < 2ac\), prove that the cubic equation

\[x^3 + ax^2 + bx + c = 0\]

cannot have all real roots.

Problem 5: Prove

\[\sin 10 \sin 30 \sin 50 \sin 70 = \frac{1}{16}.\]

Problem 6: Three points are taken at random on a unit sphere.

What is the probability that the area of the spherical triangle exceeds the area of a great circle?