THE FIFTH ANNUAL HERZOG PRIZE EXAMINATION
November 12, 1977

Problem 1: (D. Moran) Let M be an \( n \times n \) matrix of integers whose inverse is also a matrix of integers. Prove that the number of odd entries in M is at least \( n \) and at most \( n^2 - n + 1 \).

Problem 2: (A.M.M.E 1297) Having chosen \( 0 < a_1, b_1 < 1 \) define recursively

\[
a_{n+1} = a_1(1 - a_n - b_n) + a_n
\]
and
\[
b_{n+1} = b_1(1 - a_n - b_n) + b_n.
\]
Prove that \( \lim_{n \to \infty} a_n \) and \( \lim_{n \to \infty} b_n \) both exist and evaluate these limits.

Problem 3: (L. Kelly) Consider the binary homogeneous quadratic form \( x^2 + bxy + y^2 \). Suppose it is known that this form produces perfect integral squares for all positive \( a \neq \epsilon \) choices of \( x \) and \( y \). Prove that \( b \) must be \( \pm 2 \).

Problem 4: (L. Kelly) A solid sphere rolls on a plane \( \pi \) always touching a fixed line \( L \). Find the locus of its center.

Problem 5: (L. Kelly) Show that if all the distances between pairs of points of a seven point subset of the unit disc are at least 1, then the points must be the vertices of a regular inscribed hexagon and the center of the circle.

Problem 6: (A.M.M.E 1342) If \( x, y > 0 \), prove that
\[
x^y + y^x > 1.
\]