THE THIRTEENTH ANNUAL HERZOG PRIZE EXAMINATION

November 9, 1985

Problem 1: Suppose
\[ S = f(1) + f(2) + f(3) + \cdots \]
where \( f(mn) = f(m)f(n) > 0 \) for all \( m \) and \( n \). Compute
\[ f(1) + f(3) + f(5) + f(7) + \cdots , \]
\[ f(2) + f(4) + f(6) + f(8) + \cdots , \]
and
\[ f(1) - f(2) + f(3) - f(4) + f(5) - \cdots . \]

Problem 2: (M.J. Winter) What is the expected number of throws of a fair coin until heads occur twice in succession.

Problem 3: (J.G. Hocking) Given two circles and a point \( P \) on one, construct a circle through \( P \) tangent to both given circles.

Problem 4: (Wade C. Ramey) Let \( P(x,y) \) be a polynomial where
\[ \int_D \int P(x,y) \, dx \, dy = 0 \]
over every disk \( D \) of radius 1 containing the origin. Prove that \( P(x,y) \) is the 0 polynomial.

Problem 5: (L.M. Kelly) Is the volume of a tetrahedron a function of the areas of its (four) faces?

Problem 6: (J.G. Hocking) Suppose \( f \) is differentiable on \([a,b]\) with \( f(a) = a \) and \( f(b) = b \). Do there exist \( a < x_1 < x_2 < b \) with
\[ \frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2 ? \]