THE TWENTY-FIRST HERZOG PRIZE EXAMINATION
November 13, 1993

1. Compute the length of an edge of a regular pentagon inscribed in a circle of unit radius. (Solve in radicals.)

2. Are there any angles $\theta$ for which $\sqrt{\sin \theta}$ and $\sqrt{\cos \theta}$ are both non-zero rational numbers?

3. Find all positive integers $m$ and $n$ for which
   \[ 1! + 2! + \cdots + n! = m^2. \]

4. In a given tetrahedron the sum of the angles at each of its vertices is $180^\circ$. Prove that all the faces of the tetrahedron are congruent.

5. Show that the equation
   \[ a^2x^n = x^{n-1} + x^{n-2} + \cdots + x + 1 \]
   has exactly one positive real solution.

6. Assume that $2 \leq x \leq y$. Prove or disprove that $y^{x+1} \leq xy^x$. 