Pullback diagrams of relative Toeplitz graph algebras

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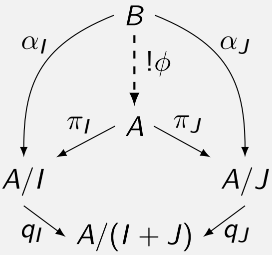
Pullbacks of quotients

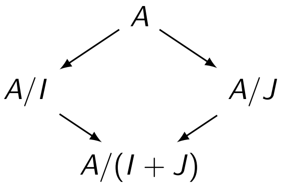
Theorem 1 (Pedersen)

Let B be a C^*-algebra and let $\alpha_I : B \to A/I$ and $\alpha_J : B \to A/J$ be $*$-homomorphisms such that $q_I \circ \alpha_I = q_J \circ \alpha_J$. If $IJ = \{0\}$ then there exists a unique $*$-homomorphism $\phi : B \to A$ such that the following diagram commutes.



Pullbacks of quotients

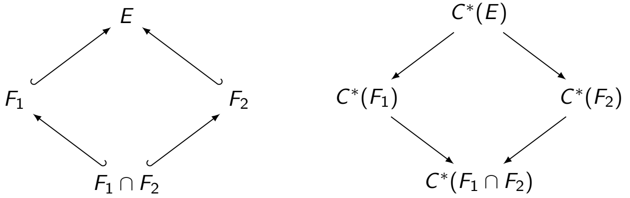
Summary: If I and J are ideals of a C^*-algebra A, then



is a pullback diagram of C^*-algebras if and only if $IJ(= I \cap J) = \{0\}$.

Motivation

In a recent paper [3], Hajac, Reznikoff, and Tobolski provide conditions they call *admissibility* on a decomposition of a directed graph E into a pair of subgraphs (F_1, F_2) that imply that the *Cuntz-Krieger graph C^*-algebras* of the three graphs fit into a pullback diagram that is dual to the pushout diagram of the underlying graphs:



Pullback diagrams of relative Toeplitz graph algebras

S. Brooker

Introduction

Graph Algebras

Graphs

Paths

Various graph algebras

Ideals

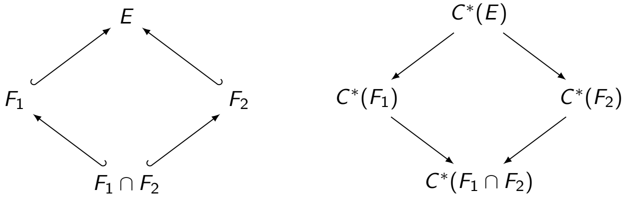
The graph groupoid

$G(E)$

Groupoids and ideals

Relative graphs, pushouts, and pullbacks

The algebras of the subgraphs considered in [3] are quotients of $C^*(E)$ by *gauge-invariant ideals.* However, such quotients cannot always be realized as Cuntz-Krieger algebras of subgraphs.



Spielberg introduced *Relative Toeplitz graph algebras* to describe subalgebras corresonding to subgraphs [9], but they also arise as *quotients* by gauge-invariant ideals that correspond to subgraphs.

Graphs

Definition 2

A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$

- E^0 is the set of vertices
- E^1 is the set of directed edges
- $r, s : E^1 \to E^0$ are the range and source maps, respectively, so that if $e \in E^1$ is an edge from v to w

$$w \xleftarrow{\;e\;} v$$

then $r(e) = w$ and $s(e) = v$.

Notation: For $v \in E^0$, write

$$vE^1 = \{e \in E^1 : r(e) = v\} = r^{-1}(v)$$

and similarly, E^1v for $s^{-1}(v)$.

Paths

Definition 3

- Denote by E^* the collection of finite paths in E: $\alpha \in E^*$ means $\alpha = e_1 e_2 \cdots e_n$, where $e_1, \dots, e_n \in E^1$ and $s(e_i) = r(e_{i+1})$ for $1 \leq i < n$:

$$r(\alpha) := r(e_1) \xleftarrow{e_1} \xleftarrow{e_2} \xleftarrow{\dots} \xleftarrow{e_n} s(e_n) =: s(\alpha)$$

- Let E^∞ be the set of all (semi) infinite paths in E: $x \in E^\infty$ means that $x = e_1 e_2 \cdots$ where $s(e_i) = r(e_{i+1})$ for $i \geq 1$:

$$r(x) := r(e_1) \xleftarrow{e_1} \xleftarrow{e_2} \xleftarrow{e_3} \cdots$$

For $\alpha \in E^*$, write $\alpha E^* = \{\alpha\beta : \beta \in E^*, r(\beta) = s(\alpha)\}$. The sets $E^*\alpha$, αE^∞, and $E^* x$ for $x \in E^\infty$ are defined similarly.

A graph algebra: $\mathcal{T}C^*(E)$

Definition 4

Let E be a (directed) graph. The *Toeplitz graph algebra* $\mathcal{T}C^*(E)$ is the universal C^*-algebra generated by a set of mutually orthogonal projections $\{P_v : v \in E^0\}$ and a set of partial isometries $\{S_e : e \in E^1\}$ satisfying the following relations:

- for all $e \in E^1$, $S_e^* S_e = P_{s(e)}$
- for all $e, f \in E^1$, if $e \neq f$ then $S_e^* S_f = 0$
- for all $e \in E^1$, $P_{r(e)} S_e = S_e$

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...But this is not the algebra people are usually referring to when they talk about "graph C^*-algebras."

The graph algebra, $C^*(E)$

Definition 5

A vertex v in a graph E is called *singular* if it is a *source*, $|vE^1| = 0$, or an *infinite receiver*, $|vE^1| = \infty$. Otherwise, v is called *regular*. We write $\mathrm{reg}(E)$ for the set of regular vetices in E.

Definition 6

Let E be a graph. The *Cuntz-Krieger algebra of E*, denoted $C^*(E)$, is defined as $\mathcal{T}C^*(E)$, but with the additional requirement that for all $v \in \mathrm{reg}(E)$, $P_v = \sum_{e \in vE^1} S_e S_e^*$.

The relation

$$P_v = \sum_{e \in vE^1} S_e S_e^*$$

is called the *Cuntz-Krieger relation at v*.

One more graph algebra

Definition 7

Let E be a graph, and let $A \subseteq \mathrm{reg}(E)$. The *relative Toeplitz graph algebra of E given by A*, denoted $\mathcal{T}C^*(E, A)$, is defined as $\mathcal{T}C^*(E)$, but with the additional requirement that for all $v \in A$, $P_v = \sum_{e \in vE^1} S_e S_e^*$.

Note that $\mathcal{T}C^*(E) = \mathcal{T}C^*(E, \emptyset)$, and $C^*(E) = \mathcal{T}C^*(E, \mathrm{reg}(E))$.

Gauge-invariant ideals of graph algebras

Definition 8

- Let E be a graph. A set $H \subseteq E^0$ is called *hereditary* if for any $e \in HE^1$, $s(e) \in H$.
- For a hereditary set $H \subseteq E^0$, we define a subgraph F_H of E by $F_H^0 = E^0 \setminus H$, $F_H^1 = E^1 F_H^0$.

Note that $F_H^1 = E^1 F_H^0 = F_H^0 E^1 F_H^0$, since H is hereditary.

Theorem 9 [8]

The gauge-invariant ideals of $\mathcal{T}C^*(E)$ are parameterized by pairs (H, A) where H is a hereditary subset of E^0 and $A \subseteq reg(F_H)$.

Groupoids

Definition 10

A *groupoid* is a small category G in which every morphism has an inverse. Denote by $G^{(0)}$ the set of objects (identified with their identity morphisms), called *units*. For a general element $g \in G$, define range and source maps $r, s : G \to G^{(0)}$ by

$$r(g) = gg^{-1}, \quad s(g) = g^{-1}g.$$

Given a topological groupoid G, the (full) groupoid C^*-algebra $C^*(G)$ is built from *-representations of the convolution *-algebra $C_C(G)$ (see [5] for details).

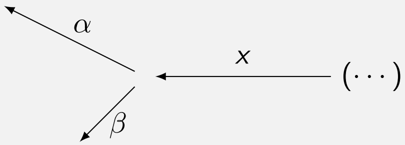
The groupoid of a graph

We use the description from [10]. This is not the usual description.

Definition 11

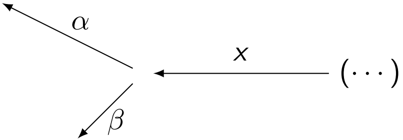
Let E be a directed graph. The *groupoid of* E, $G(E)$, is defined as follows:

- The unit space is $G(E)^{(0)} = E^\infty \cup E^*$, the set of all infinite and finite paths in E.
- The elements of $G(E)$ are equivalence classes of triples, written $[\alpha, \beta, x]$, where $\alpha, \beta \in E^*$, $x \in G(E)^{(0)}$, and $s(\alpha) = s(\beta) = r(x)$.



What is $[\alpha, \beta, x] \in G(E)$?

Let $t = [\alpha, \beta, x] \in G(E)$.



Then $r(t) = [r(\alpha), r(\alpha), \alpha x] \equiv \alpha x$ and $s(t) = \beta x \in G(E)^{(0)}$.

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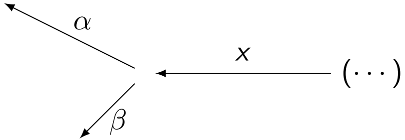

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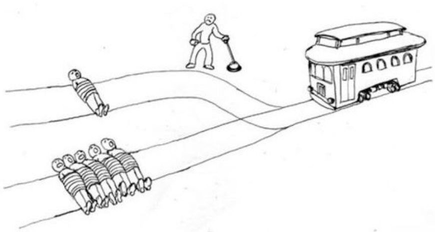
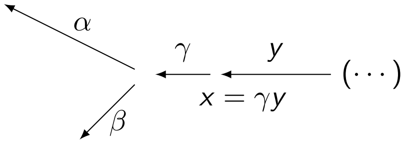


Equivalence classes

Let $t = [\alpha, \beta, x] \in G(E)$, where $x = \gamma y$.



Then $(\alpha, \beta, x) = (\alpha, \beta, \gamma y) \sim (\alpha\gamma, \beta\gamma, y)$.

So, $t = [\alpha, \beta, x] = [\alpha\gamma, \beta\gamma, y]$.

The groupoid of a graph: topology

For $\alpha \in E^*$, let $Z(\alpha) = \alpha G(E)^{(0)}$. Then sets of the form $Z(\alpha) \setminus \bigcup_1^n Z(\beta_i)$ form a basis of compact-open sets for a totally disconnected, locally compact Hausdorff topology on the unit space.

This induces a basis of compact-open bisections for an LCH topology on the whole groupoid, with respect to which $G(E)$ is an étale, amenable, locally compact, Hausdorff groupoid.

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In [10] it is proven that $C^*(G(E)) \cong \mathcal{T}C^*(E)$. We use the groupoid picture to work with the gauge-invariant ideals of $\mathcal{T}C^*(E)$ and their quotients.

Groupoids and ideals

Definition 12

Let G be a groupoid. A *G-invariant* (or just *invariant*) *set* is a subset $S \subseteq G^{(0)}$ such that

$$\forall t \in G, r(t) \in S \iff s(t) \in S.$$

For an invariant set S, we write S^C for the relative complement $G^{(0)} \setminus S$ of S in $G^{(0)}$. Note that $S \subseteq G^{(0)}$ is invariant if and only if S^C is invariant.

The gauge-invariant ideals of $\mathcal{T}C^*(E) = C^*(G(E))$ are given by the open $G(E)$-invariant sets $U \subseteq G(E)^{(0)}$: Given U, $G(E)|_U$ is a groupoid whose unit space is U, and we have the following short exact sequence:

$$0 \to C^*(G(E)|_U) \to C^*(G(E)) \to C^*(G(E)|_{U^C}) \to 0$$

Connecting the dots

Earlier we said that the gauge-invariant ideals of $\mathcal{T}C^*(E)$ correspond to pairs (H, A) where $H \subseteq E^0$ is hereditary, and $A \subseteq \text{reg}(F_H)$. Thus, each pair (H, A) corresponds to an open invariant set $U(H, A) \subseteq G(E)^{(0)}$.

We give a description of the set $U(H, A)$ in [2], and get the following:

Theorem 13 [2] (B-Spielberg 2022)

If $J_{H,A}$ is the ideal $C^*(G(E)|_{U(H,A)})$ of $C^*(G(E)) = \mathcal{T}C^*(E)$ corresponding to a hereditary set $H \subseteq E^0$ and a subset $A \subseteq \text{reg}(F_H)$, then

$$\mathcal{T}C^*(E)/J_{H,A} \cong \mathcal{T}C^*(F_H, A).$$

Results

We can use relationships between the open invariant subsets of the unit space (and their closed invariant complements) to characterize relationships between ideals and quotients of $\mathcal{T}C^*(E)$.

Then we can realize these unit space relationships in terms of pairs of the form (H, A), and state results entirely in the language of graphs and graph algebras.

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Theorem 14 [2] (B-Spielberg 2022)

Let E be a graph, $H \subseteq E^0$ a hereditary set, and $F = F_H$. Let $A \subseteq reg(E)$ and $B \subseteq reg(F)$. Then $\mathcal{T}C^*(F, B)$ is the quotient of $\mathcal{T}C^*(E, A)$ by a gauge-invariant ideal if and only if $A \cap F^0 \subseteq B$.

Relative graphs

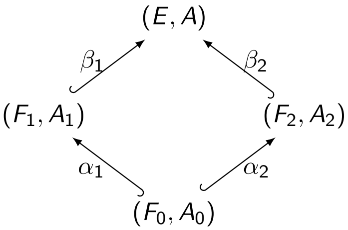
In [2] we introduce a category of *relative graphs,* in which an object is a pair (F, B) consisting of a directed graph F and a set $B \subseteq \text{reg}(F)$, and a morphisim $\alpha : (F, B) \to (E, A)$ is an inclusion of graphs $F \hookrightarrow E$ satisfying the conditions under which $\mathcal{T}C^*(F, B)$ is isomorphic to the quotient of $\mathcal{T}C^*(E, A)$ by an ideal that we denote by $J(E, A; F, B)$.

Pushouts of relative graphs

Theorem 15 [2] (B-Spielberg 2022)

The category of relative graphs admits pushouts: If $\alpha_i : (F_0, A_0) \to (F_i, A_i)$ *are relative graph morphisms for* $i = 1, 2$, *then define* (E, A) *by*

$$E = F_1 \sqcup_{F_0} F_2,$$
$$A = (A_1 \setminus F_0^0) \cup (A_2 \setminus F_0^0) \cup (A_1 \cap A_2).$$



The main result: We get pullbacks!

$$
\begin{array}{ccccccc}
\mathcal{T}C^*(E,A) & \longrightarrow & \dfrac{\mathcal{T}C^*(E,A)}{J(E,A;F_1,A_1)} & \xrightarrow{\cong} & \mathcal{T}C^*(F_1,A_1) & & \\
\downarrow & & & & \downarrow & & \\
\dfrac{\mathcal{T}C^*(E,A)}{J(E,A;F_2,A_2)} & & & & \dfrac{\mathcal{T}C^*(F_1,A_1)}{J(F_1,A_1;F_0,A_0)} & & \\
\cong\downarrow & & & & \downarrow\cong & & \\
\mathcal{T}C^*(F_2,A_2) & \longrightarrow & \dfrac{\mathcal{T}C^*(F_2,A_2)}{J(F_2,A_2;F_0,A_0)} & \xrightarrow{\cong} & \mathcal{T}C^*(F_0,A_0) & \overset{\cong}{=\!=} & \dfrac{\mathcal{T}C^*(E,A)}{J(E,A;F_0,A_0)}
\end{array}
$$

Theorem 16 [2] (B-Spielberg 2022)

Given the pushout (E,A) of relative graphs $(F_1,A_1),(F_2,A_2)$ over (F_0,A_0) as in Theorem 15, the corresponding commuting square of relative Toeplitz graph C^*-algebras is a pullback diagram if and only if

$$A_0 \subseteq A_1 \cup A_2.$$

Thank You! I

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