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Properly proximal von Neumann algebras

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$\Gamma \rightsquigarrow$ group vNa $L\Gamma$

$\Gamma \curvearrowright (X, \mu)$ p.m.p. free ergodic action $\rightsquigarrow$ group measure space $L^\infty(X) \rtimes \Gamma$

distinguish these vNas by the inital data,

study structure : e.g. $L\Gamma$ is prime, i.e. $L\Gamma \not\cong M_1 \bar{\otimes} M_2$, $\dim(M_i) = \infty$

has no Cartan subalgebra

$L^\infty(X) \rtimes \Gamma$ has $L^\infty(X)$ as its unique Cartan, up to unitary conjugacy

(reduce the classification to OE of $\Gamma \curvearrowright X$)
$\uparrow$ relation

Ozawa's Solidity theorem '03 : If $\Gamma = \mathbb{F}_n$,

then $L\Gamma$ is solid (any diffuse subalgebra $A < L\Gamma$ has amenable relative commutant $A' \cap L\Gamma$) and hence $L\mathbb{F}_n$ is prime for $n \geq 2$.

this is true for all biexact groups

Γ is biexact if $\Gamma \curvearrowright \mathbb{S}(\Gamma)$ is topological amenable action

- $\mathbb{S}(\Gamma) = \{ f \in \ell^\infty\Gamma \mid f - R_t f \in c_0\Gamma, \forall t \in \Gamma \} \subset \ell^\infty\Gamma$.
- action : restriction of left action on $\ell^\infty\Gamma$
- amenable: $\mathbb{S}(\Gamma) \rtimes_r \Gamma$ is nuclear C^*-algebra.

main examples: hyperbolic groups

(Ozawa - Popa '07. Chifan - Sinclair '11) Γ non amen biexact weakly amen. then $L\Gamma$ has no Cartan

On the group measure side:

(Popa - Vaes '11) Γ non amen biexact weakly amen. then any free ergodic p.m.p action $\Gamma \curvearrowright (X, \mu)$, $L^\infty(X) \rtimes \Gamma$ has a unique Cartan

$(\forall\ \mathbb{F}_n \curvearrowright (X, \mu),\ \mathbb{F}_m \curvearrowright (Y, \nu),\ \ L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m \Rightarrow n = m)$

$\mathbb{F}_n \curvearrowright X \sim_{OE} \mathbb{F}_m \curvearrowright Y$

In Boutonnet - Ioana - Peterson '18, the Biexact method is further generalized.

Γ is properly proximal if $\nexists$ Γ-inv state on $S(\Gamma)$.

(Boutonnet - Ioana - Peterson '18) $\Gamma \curvearrowright (X, \mu)$ free ergodic p.m.p. weakly compact ($\exists\ \varphi: \mathbb{B}(L^2(X)) \to \mathbb{C}$ state $L^\infty(X)$ central and Γ-inv) Γ is prop prox, then $L^\infty(X) \rtimes \Gamma$ has $L^\infty(X)$ as its unique weakly compact Cartan

Examples:
- nonamenable biexact
- nonelementary relative hyperbolic groups
- higher rank lattices $SL_n(\mathbb{Z})$, $n \geq 3$

} BIP '18

- mapping class groups. Horbez, Huang, Lecureux '20
- $\Lambda \wr \Gamma$, Λ non-trivial, Γ non-amen DKE '22.
- any group ME, W*E to above groups Ishan Peterson Ruth '19.

Non-example: ME class of inner amenable

Γ prop prox $L\Gamma \cong L\Lambda \Rightarrow \Lambda$ is prop prox

definition for vNa.

Γ is prop prox if $\nexists$ Γ- inv state on

$$S(\Gamma) = \{ f \in \ell^\infty \Gamma \mid f - R_t f \in C_0 \Gamma \quad \forall t \in \Gamma \}.$$

M a finite vNa

$S^0(M) = \{ T \in \mathbb{B}(L^2M) \mid [T, x] \in \mathbb{K}(L^2M), \forall x \in JMJ \}$

but this is derivations from $JMJ \longrightarrow \mathbb{K}(L^2M)$ which are all inner

i.e. $S^0(M) = M + \mathbb{K}(L^2M)$

fix: consider $\overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}} \subset \mathbb{B}(L^2M)$, $\|\cdot\|_{\infty,1}$ is the operator norm

by thinking $T \in \mathbb{B}(L^2M)$ as $T : (\hat{M}, \|\cdot\|_\infty) \longrightarrow L^2M \subset (L^1M, \|\cdot\|_1)$

$$\|T\|_{\infty,1} = \sup_{a,b \in (M)_1} \langle T\hat{a}, \hat{b} \rangle.$$

- equivalently, $T \in \overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}}$ iff $\exists$ $p_n \in \mathcal{P}(M)$, $p_n \nearrow 1$ s.t.

  $p_n Jp_nJ \, T \, Jp_nJ \, p_n \in \mathbb{K}(L^2M)$,

- abstract operator space description due to Magajna.

$S(M) = \{ T \in \mathbb{B}(L^2M) \mid [T, x] \in \overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}}, \forall x \in JMJ \}$.

M is prop prox if $\nexists$ M- central state φ on $S(M)$ s.t. $\varphi|_M$ is normal.

Thm (D. Kunnawalkam - Elayavalli, Peterson '22) Γ is prop prox iff $L\Gamma$ is prop prox.

$\ell^\infty \Gamma \hookrightarrow \mathbb{B}(\ell^2\Gamma)$

restriction

$S(\Gamma) \hookrightarrow S(L\Gamma)$

density property of $\overline{\mathbb{K}(L^2M)}^{\|\cdot\|_{\infty,1}}$

$E : \mathbb{B}(\ell^2\Gamma) \longrightarrow \ell^\infty\Gamma$

restriction

$S(L\Gamma) \longrightarrow S(\Gamma)$

E is $\|\cdot\|_{\infty,1}$ to norm continuous

Γ biexact, then any nonamen $\Lambda < \Gamma$ is not inner amen

any nonamen $N \subset L\Gamma$ has no (Gamma)

Popa asks if $L\Sigma \hookrightarrow L\Gamma$, Σ nonamen inner amen.

Thm: Γ biexact, $N \subset L\Gamma$ a subalg. Then either N is amen or N is prop prox. In particular, $L\Sigma \not\hookrightarrow L\Gamma$.

Ex: $M_1 * M_2$, M_i diffuse

$L\Gamma$, Γ prop prox

$L^\infty(X) \rtimes \Gamma$, $\Gamma \curvearrowright (X, \mu)$ Gaussian action from nonamen rep

Γ prop prox.