Cuntz-Pimsner algebras

arising from

C*-correpondences over commutative C*-algebras

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$$\mathrm{Ell}(\mathfrak{A}) = (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A}), T(\mathfrak{A}), \rho)$$

Theorem (Classification theorem)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, infinite-dimensional, unital, simple C^*-algebras with finite nuclear dimension and which satisfy the UCT. Suppose there is an isomorphism

$$\psi : \mathrm{Ell}(\mathfrak{A}) \to \mathrm{Ell}(\mathfrak{B}).$$

Then there is a *-isomorphism

$$\Psi : \mathfrak{A} \to \mathfrak{B},$$

which is unique up to approximate unitary equivalence and satisfies $\mathrm{Ell}(\Psi) = \psi$.

Definition

A (right) Hilbert $\mathfrak{A}$-module is a right $\mathfrak{A}$-module $\mathcal{E}$ equipped with an $\mathfrak{A}$-valued inner product such that $\mathcal{E}$ is complete in the norm $\|\xi\|_{\mathcal{E}} = \|\langle\xi,\xi\rangle_{\mathcal{E}}\|_{\mathfrak{A}}^{1/2}$.

Definition

A C*-correspondence over $\mathfrak{A}$ is a right Hilbert $\mathfrak{A}$-module $\mathcal{E}$ equipped with a structure map $\varphi_{\mathcal{E}} : \mathfrak{A} \to \mathcal{L}(\mathcal{E})$.

Definition

A Hilbert $\mathfrak{A}$-bimodule is a right and left Hilbert $\mathfrak{A}$-module which satisfies the compatibility condition

$$ {}_{\mathcal{E}}\langle\xi,\eta\rangle \cdot \zeta = \xi \, \langle\eta,\zeta\rangle_{\mathcal{E}} $$

$\mathcal{E} = \mathfrak{A}$ is a C*-correspondence over $\mathfrak{A}$:

$$\langle a, b\rangle_{\mathcal{E}} = a^* b$$

$$\varphi : \mathfrak{A} \to \mathcal{L}(\mathcal{E}) \text{ given by } a \mapsto (b \mapsto ab).$$

The left inner product ${}_{\mathcal{E}}\langle a, b\rangle = ab^*$ makes $\mathcal{E}$ into a Hilbert $\mathfrak{A}$-bimodule.

Definition (Katsura, 2004)

Let A be a C^*-algebra and let $\mathcal{E}$ be a C^*-correspondence over $\mathfrak{A}$. The *Cuntz–Pimsner algebra of $\mathcal{E}$ over $\mathfrak{A}$*, denoted $\mathcal{O}(\mathcal{E})$, is the C^*-algebra generated by the universal covariant representation of $\mathcal{E}$.

Example

X - compact metric space

$\mathcal{V} = [V, p, X]$ a vector bundle

- locally trivial: At every x there exists a neighbourhood U of x such that $\mathcal{V}|_U \cong U \times \mathbb{C}^{n_x}$.

Definition

rank$(\mathcal{V}) = n$ if $p^{-1}(x) \cong \mathbb{C}^n$ for every $x \in X$.

Definition

continuous sections $\Gamma(\mathcal{V}) := \{$continuous maps $\xi : X \to \mathcal{V} : p(\xi(x)) = x\}$

- $\Gamma(\mathcal{V})$ is a right $C(X)$-module via

$$(\xi \cdot f)(x) = \xi(x) f(x).$$

- use charts and a partition of unity on X to construct a $C(X)$-valued inner product

Theorem

$\mathcal{E} = \Gamma(\mathcal{V})$ for $\mathcal{V}$ a complex line bundle over X.
$\alpha : X \to X$ homeomorphism

Define

$$f \cdot \xi := \xi f \circ \alpha \qquad f \in C(X), \xi \in \mathcal{E},$$
$$_{\mathcal{E}}\langle \xi, \eta \rangle := \langle \eta, \xi \rangle_{\mathcal{E}} \circ \alpha^{-1} \qquad \xi, \eta \in \mathcal{E}$$

to make $\mathcal{E}$ into a Hilbert $C(X)$-bimodule, which we denote by $\Gamma(\mathcal{V}, \alpha)$.

Definition

A correspondence $\mathcal{E}$ is full if $\text{span}\{\langle \xi, \eta \rangle_{\mathcal{E}} : \xi, \eta \in \mathcal{E}\}$ is dense in $\mathfrak{A}$.

If $\mathcal{E}$ is a bimodule then we can talk about right full or left full depending on which inner product we reference. In the above example, $\mathcal{E}$ is both left and right full.

If $\mathcal{V}$ is not a line bundle we can perform a similar construction but we get a C*-correspondence which is not a Hilbert bimodule.

X be a compact metric space
$\mathcal{V} = [V, p, X]$ a vector bundle over X
$\alpha : X \to X$ a homeomorphism.

$\Gamma(\mathcal{V},\alpha)$ has the same right Hilbert $C(X)$-module structure as $\Gamma(\mathcal{V})$ and structure map $\varphi : C(X) \to \mathcal{K}(\Gamma(\mathcal{V},\alpha))$ defined by

$$\varphi(f)(\xi) = \xi f \circ \alpha.$$

Theorem (Serre–Swan)

Let X be a compact metric space and $\mathcal{E}$ be an algebraically finitely generated projective right $C(X)$-module. Then there exists a vector bundle $\mathcal{V} = [V, p, X]$ such that $\mathcal{E} \cong \Gamma(\mathcal{V})$ as right $C(X)$-modules.

If $\mathcal{V}$ is a line bundle and $\mathcal{E}$ is a full $C(X)$-bimodule (as both a left and a right module) then there exists a homeomorphism $\alpha : X \to X$ such that $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$ (this is a result of Abadie and Exel).

For $\mathcal{V}$ a trivial line bundle over X, $\mathcal{O}(\Gamma(\mathcal{V},\alpha)) \cong C(X) \rtimes_\alpha \mathbb{Z}$.

For $\mathcal{V}$ a line bundle and $\alpha = \mathrm{id}$, $\mathcal{O}(\Gamma(\mathcal{V})) \cong C(X) \rtimes_{\Gamma(\mathcal{V})} \mathbb{Z}$.

- example: quantum Heisenberg manifolds

Separable, unital and infinite-dimensional follow easily from the construction.

Theorem (Katsura 2004)

$\mathcal{O}(\mathcal{E})$ is simple if and only if $\mathcal{E}$ is minimal and nonperiodic.

Theorem

$\mathcal{O}(\Gamma(\mathcal{V},\alpha))$ is simple if and only if α is minimal.

Theorem (Brown-Tikuisis-Zelenberg 2018)

Suppose $\mathfrak{A}$ is separable and unital and $\mathcal{E}$ is a finitely generated projective C-correspondence over $\mathfrak{A}$. If $\mathfrak{A}$ has finite nuclear dimension and $\mathcal{E}$ has finite Rokhlin dimension then $\mathcal{O}(\mathcal{E})$ has finite nuclear dimension.*

Theorem

$\mathcal{V} = [V, p, X]$ *with* $\dim(X) < \infty$ *and* $\alpha : X \to X$ *aperiodic*
$\mathcal{O}(\Gamma(\mathcal{V},\alpha))$ *has finite nuclear dimension.*

Theorem (Katsura 2004)

Suppose $\mathfrak{A}$ is separable and nuclear and $\mathcal{E}$ is a separable C-correspondence over $\mathfrak{A}$. If $\mathfrak{A}$ and $\mathcal{J}_{\mathcal{E}}$ satisfy the UCT then so does $\mathcal{O}(\mathcal{E})$.*

Theorem

$\mathcal{O}(\Gamma(\mathcal{V},\alpha))$ satisfies the UCT.

Theorem

Let $\mathcal{C}$ denote the class of C^*-algebras of the form $\mathcal{O}(\Gamma(\mathcal{V},\alpha))$ for X an infinite compact metric space with $\dim(X) < \infty$, $\mathcal{V} = [V,p,X]$ a vector bundle, and $\alpha : X \to X$ a minimal homeomorphism. The algebras in $\mathcal{C}$ are classifiable.

Theorem

Let $\mathcal{E} = \Gamma(\mathcal{V},\alpha)$ where $\mathcal{V} = [V,p,X]$ is a vector bundle and $\alpha : X \to X$ is a homeomorphism. Then $T(\mathcal{O}(\mathcal{E})) \neq \varnothing$ if and only if $\mathcal{V}$ is a line bundle.

Theorem

Let X be an infinite compact metric space, $\mathcal{V} = [V,p,X]$ a line bundle, and $\alpha : X \to X$ an aperiodic homeomorphism. Let $\mathcal{E} := \Gamma(\mathcal{V},\alpha)$.
Then there are affine homeomorphisms

$$T(\mathcal{O}(\mathcal{E})) \cong T(C(X) \rtimes_\alpha \mathbb{Z}) \cong M^1(X,\alpha),$$

where $M^1(X,\alpha)$ denotes the space of α-invariant Borel probability measures.

Theorem

Let $\mathfrak{A} = \mathcal{O}(\Gamma(\mathcal{V},\alpha)) \in \mathcal{C}$.

1. *If* $\mathcal{V}$ *is a line bundle,* $\mathfrak{A}$ *is stably finite.*
2. *If* $\mathcal{V}$ *has (not necessarily constant) rank greater than one,* $\mathfrak{A}$ *is purely infinite.*

$$C(X) \subset \underbrace{\text{orbit-breaking subalgebras}}_{\mathfrak{B}} \subset \underbrace{C(X) \rtimes_\alpha \mathbb{Z}}_{\mathfrak{A}}$$

unitary $u \in C(X) \rtimes_\alpha \mathbb{Z}$ such that $uf = f \circ \alpha^{-1} u$
$\mathfrak{B} := C^*(C(X), C(X \setminus Y)u)$ where Y is a closed subset of X which meets every orbit at most once.

If $\mathcal{E}$ is a C*-correspondence and $\mathcal{J}$ is an ideal of $\mathfrak{A}$ then

$$\mathcal{J}\mathcal{E} = \overline{\text{span}}\{a \cdot \xi : a \in \mathcal{J}, \xi \in \mathcal{E}\}$$

is a C*-correspondence over $\mathfrak{A}$ as well.

Consider $\mathcal{E} = \Gamma(\mathcal{V},\alpha)$ and $\mathcal{E}_Y = C_0(X \setminus Y)\mathcal{E}$.

Question: Is the subalgebra $\mathcal{O}(\mathcal{E}_Y)$ large in $\mathcal{O}(\mathcal{E})$?

Theorem

$\mathcal{V} = [V, p, X]$ a line bundle, $\alpha : X \to X$ minimal
$Y \subset X$ be a non-empty closed subset meeting each α-orbit at most once and such that for every $N \in \mathbb{Z}_{\geq 0}$ there exists an open set $W_N \supset Y$ for which $\mathcal{V}|_{\alpha^n(W_N)}$ is trivial whenever $-N \leq n \leq N$.

$\mathcal{O}(\mathcal{E}_Y)$ is a centrally large subalgebra of $\mathcal{O}(\mathcal{E})$.

Theorem

The orbit breaking algebras $\mathcal{O}(\mathcal{E}_Y)$ constructed above are classifiable.

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