ECOAS Talk Oct. 22 2022 ①

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Commuting Squares

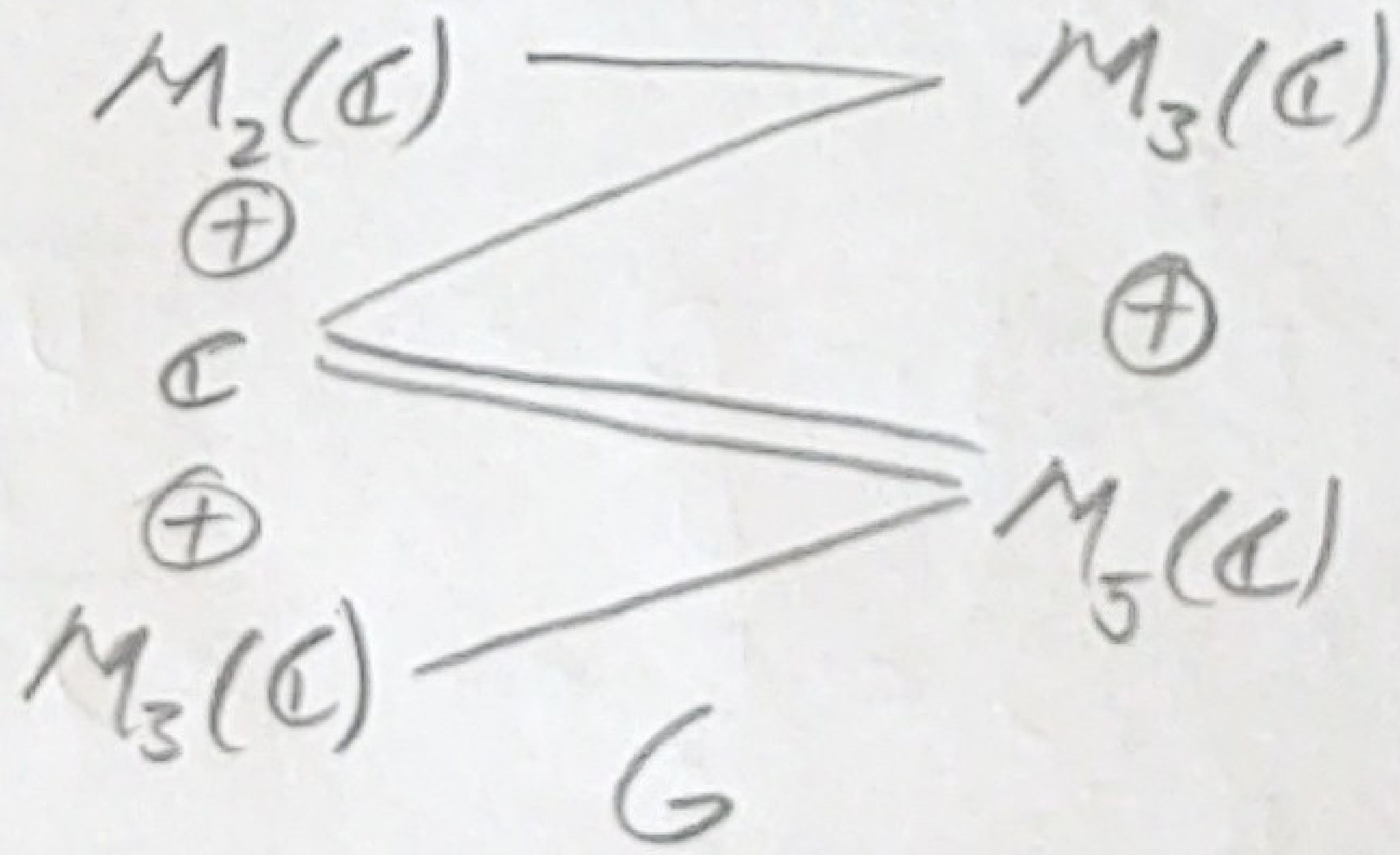
__Defn:__ A __commuting square__ is a square of f.d. C^*-alg. with a coherent trace and unital incl.

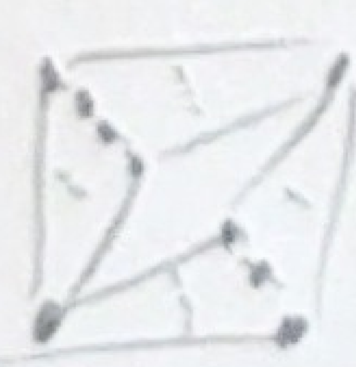
$$\begin{array}{ccc} C & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array} \quad tr,$$

s.t. $\left(L^2(C) \cap L^2(A)^{\perp}\right) \perp \left(L^2(B) \cap L^2(A)^{\perp}\right)$

up to a unitary inclusions are characterized by Bratteli diagrams

e.g.



Ex. Commuting squares of the form  are given by complex Hadamard matrices, H (has entries from $\mathbb{T} \subset \mathbb{C}$ and mut. orth. rows)

$$\begin{array}{ccc} \mathbb{C}^n & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & H\mathbb{C}^n H^* \end{array}, \quad \frac{1}{n}\mathrm{Tr}$$

Commuting squares arose naturally from the standard invariants of subfactors (initiated by Jones '83). See Popa '95 'Axiomatization of the lattice of higher relative commutants'.

<u>Defn</u>: $N \subset M$, a unital inclusion of infinite dimensional tracial vN-alg. with trivial centers is called a <u>subfactor</u>.

<u>Ex</u>:
$$\begin{array}{ccccccccccc} C & \subset & D & \subset & \langle D, e_C \rangle = D_1 & \subset & D_2 & \subset \cdots \subset & \overline{\cup D_n}^{WOT} & = & M \\ \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\ A & \subset & B & \subset & \langle B, e_C \rangle = B_1 & \subset & B_2 & \subset \cdots \subset & \overline{\cup B_n}^{WOT} & = & N \end{array}$$

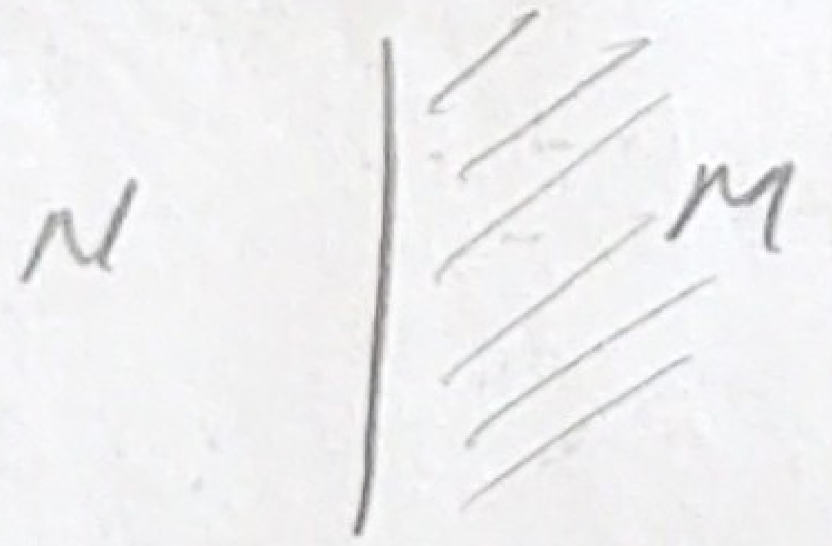
is a hyperfinite subfactor from a comm. sq if it has connected inclusions and is nondegenerate.

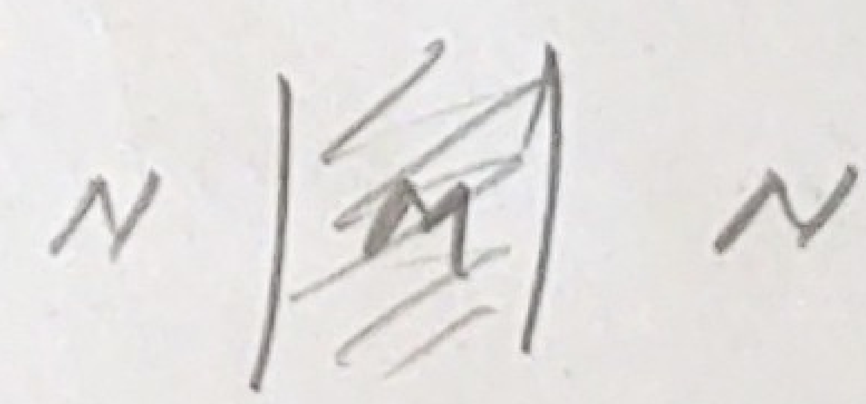
Question: What can we say about the subfactor from the initial commuting square (without too much computation). Specifically, what about it's bimodule category

Defn: $\mathcal{C} = \mathrm{bim}(N \subset M)$ is a rigid C^*-multitenso[r] category generated by the object ${}_N L^2(M)_M$ under Connes fusion, $\overline{{}_N L^2(M)_M} = {}_M L^2(M)_N$, sums $\oplus$, etc.

obj($\mathcal{C}$) ${}_N L^2(M)_M$ ${}_N L^2(M)_M \otimes_M L^2(M)_N$

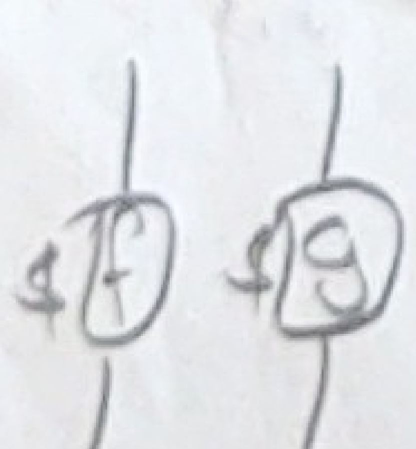
$\xi m \otimes \eta = \xi \otimes m\eta$

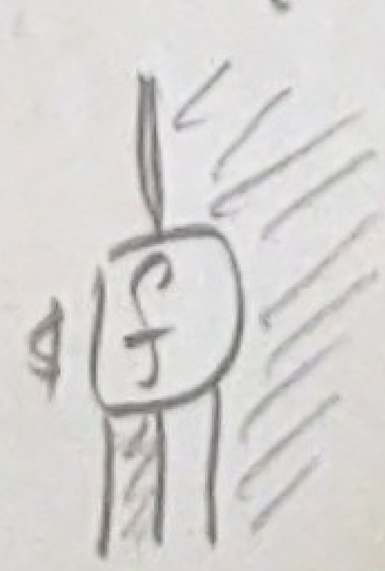




and their sub-objects.

Mor($\mathcal{C}$) ex. $f : {}_N L^2(M)_M \to {}_N L^2(M)_M \otimes_M L^2(M)_N \otimes_N L^2(M)_M$

$f \otimes g =$ 



$Obj(\mathcal{C})$ can be described by projections in $Mor(\mathcal{C})$ $\quad \alpha \to P_\alpha \in Hom({}_-L^2(M)^{\otimes k}{}_-)$

Defn: α is irreducible if $Hom(\alpha) \cong \mathbb{C}$.

Defn: The principal graph of $N \subset M$ is given by

$\alpha \in V_{Even}$ = irreducible N-N bimodules

$\beta \in V_{odd}$ = irreducible N-M bimodules

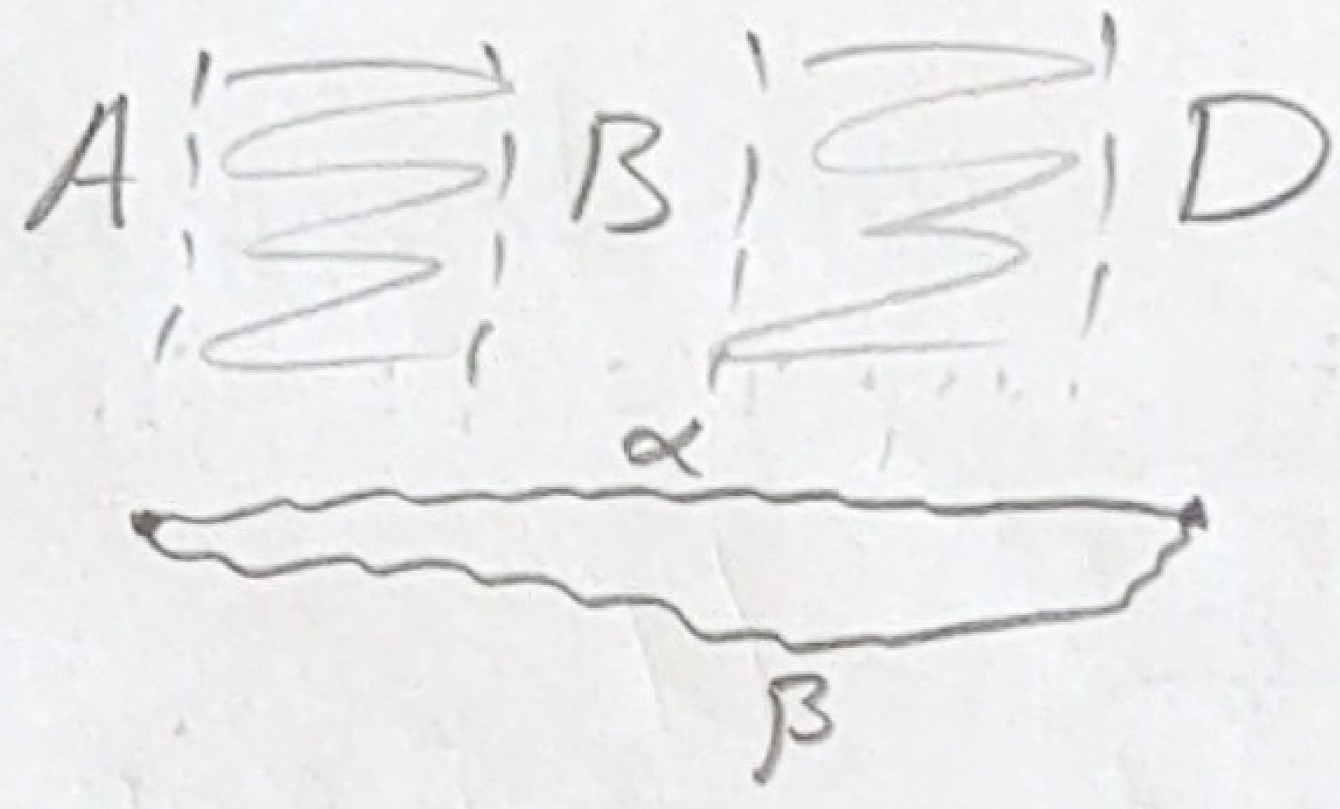
$$\# E(\alpha, \beta) = \dim(Mor(\alpha, \beta \otimes_M L^2(M)_N))$$

and is denoted Γ.

RmK: The principal graphs of a subfactor has been useful in the classification program of subfactors (see JMS'14 'The classification of subfactors of index at most 5' and AMP'15 '------ at most $5\frac{1}{4}$')

Path algebras

Given $A \subset B \subset D$ f.d. C^*-alg. Let α, β denote paths in their Bratteli diagrams



D has matrix units $\{ e_{\alpha,\beta} \mid \alpha, \beta \ A \rightsquigarrow D \text{ paths} \}$ represented on $\mathbb{C}[A \rightsquigarrow D \text{ paths}]$ by

$$e_{\alpha,\beta}\, \gamma = \delta_{\beta=\gamma}\, \alpha$$

Biunitary condition

D has two sets of matrix units

$$\begin{array}{ccc} C & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array}$$

$\{ e_{\alpha_1,\alpha_2} \mid \alpha \ A \subset B \subset D \text{ paths} \}$

$\{ f_{\beta_1,\beta_2} \mid \beta \ A \subset C \subset D \text{ paths} \}$

The square is characterized by a unitary $(U_{\alpha,\beta})$ that translates between matrix units. If the square is non-degenerate then

$$V_{\alpha,\beta} = \sqrt{\frac{t^D_\beta \, t^A_\alpha}{t^B_\alpha \, t^C_\beta}} \; U_{\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2} \qquad \begin{array}{ccc} C & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array} \;\; \beta, \; \alpha$$

is also a unitary and U is called a biunitary (This notion is due to Ocneanu '93 in "Chirality for operator algebras").

Graph Planar Algebra's (See Jones '00 "The planar alg. of a bipartite graph")

$$G = \begin{array}{ccc} \overset{\text{green}}{C} & \equiv & \overset{\text{blue}}{D} \\ \| & & \| \\ \underset{\text{yellow}}{A} & \equiv & \underset{\text{red}}{B} \end{array}$$

Vector spaces

$P^G_\partial = \mathbb{C}$[pointed loops in G with coloring ∂]

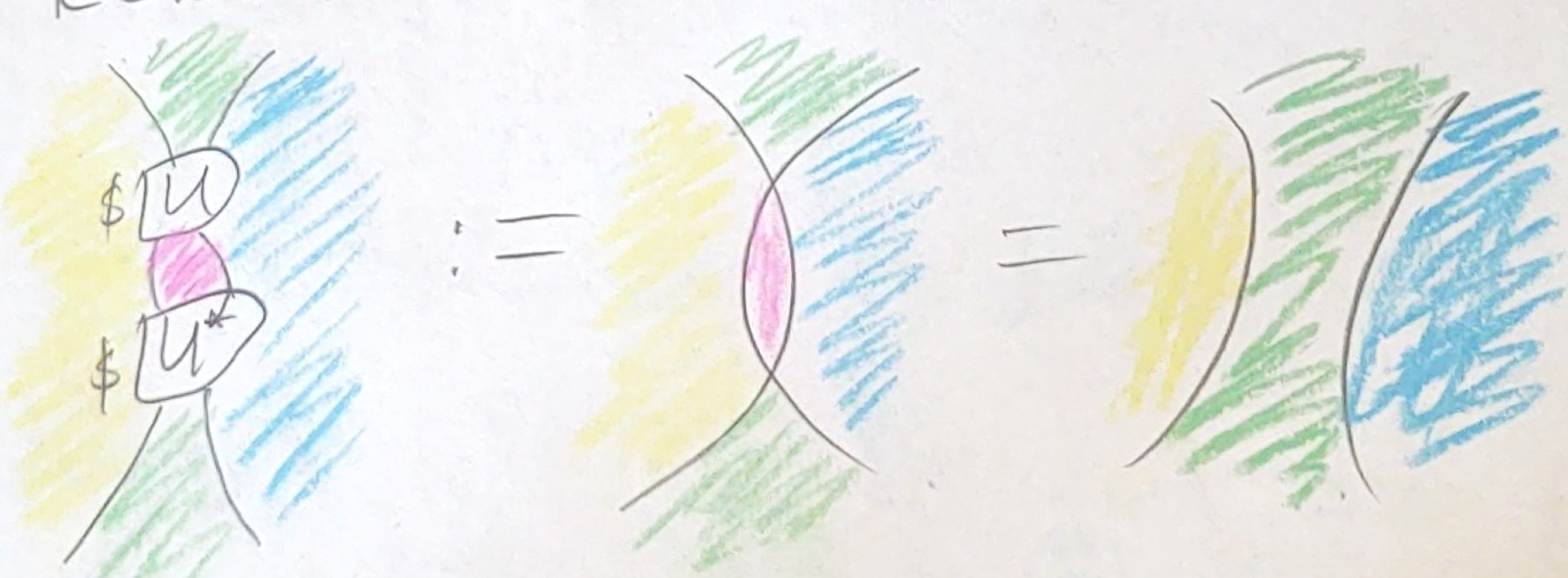
e.g. $\partial = yggg$ $A\text{-}C\text{-}A\text{-}C$ paths

Let α_i, β_i be ygyg paths then we have matrix units

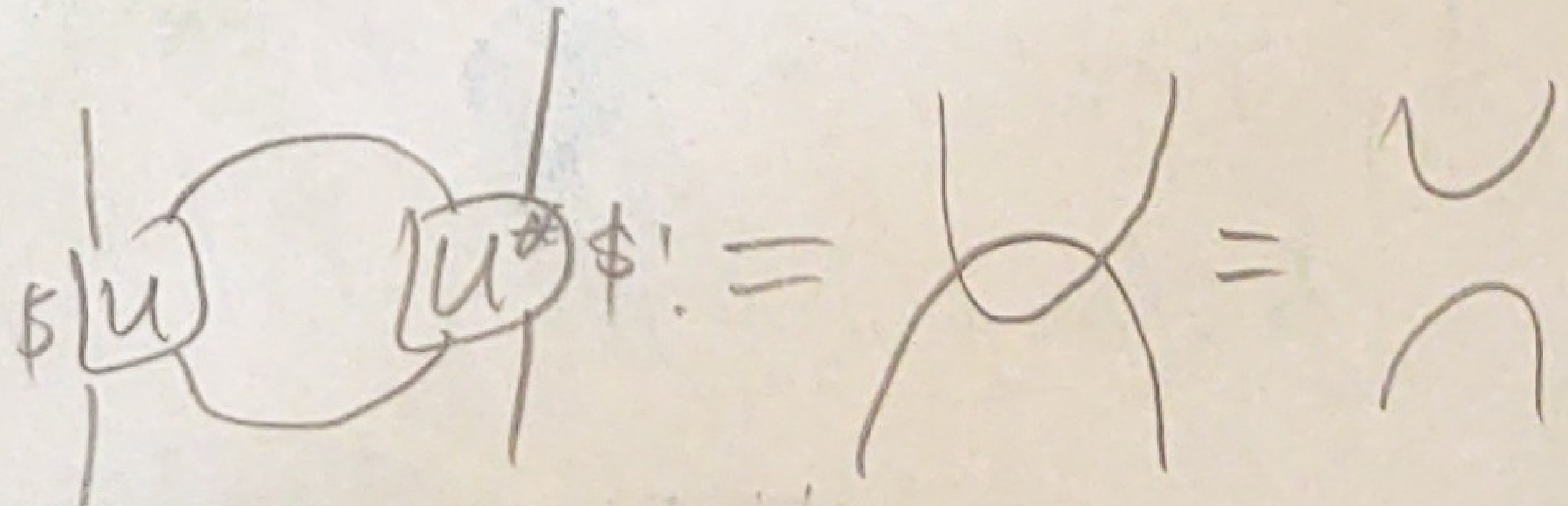
$e_{\alpha_1,\alpha_2} e_{\beta_1,\beta_2}$ =

$= \delta_{\alpha_2=\beta_1}$

The biunitary u satisfies type II Reidemeister moves of Knot theory



Similarly,



See Morrison and Peters 14 'The little desert?....'

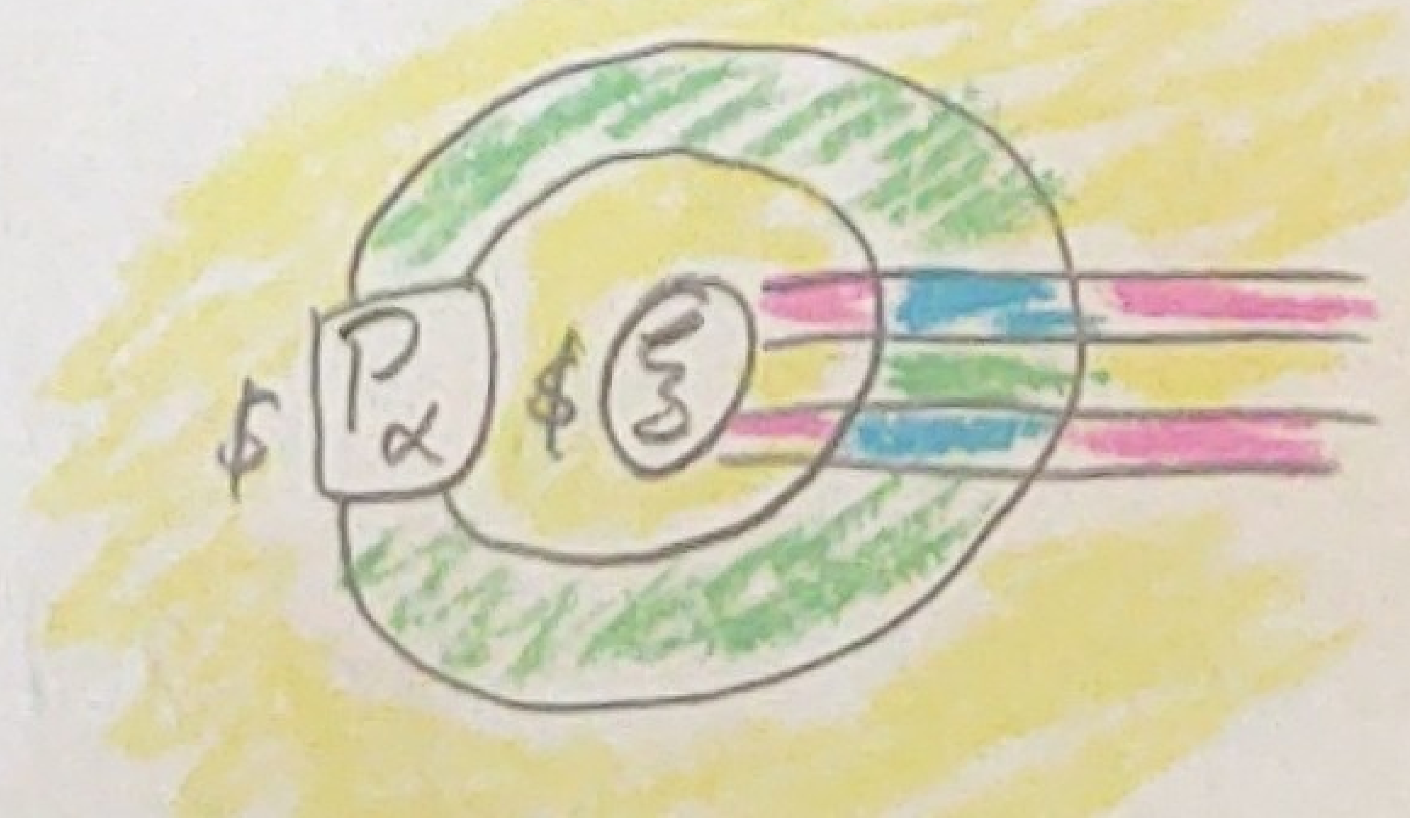
or Quan Chen '20 'Standard λ-lattices, rigid C^*-tensor categories and bimodules' for related constructions.

The morphism spaces of $\mathcal{C}$ are given by solving $x \in P^G_{(yg)^n}$ s.t. $\exists\, y \in P^G_{(rb)^n}$

as a consequence of Ocneanu compactness.
(see Jones and Penney's '11 'The embedding theorem for finite depth subfactor planar algebras' also Chen'20).

Then we may define f.d. representation of the fusion algebra $F(N \subset M)$.

$$\Phi : (\mathrm{obj}(\mathcal{C}), \otimes) \to B(P^G_{(yr)^n})$$

<u>ex.</u> $\Phi(\alpha)\xi =$ 

(See M'22 'Angle operators of commuting square subfactors') (9)

<u>Thm (M'22)</u> Φ extends to a norm preserving $*$-homomorphism of $C^*_{red}(\mathcal{F}(N \subset M)) \to B(\overline{\bigcup_n P_{(yr)^n}}^{\langle \cdot, 1 \rangle})$ iff $N \subset M$ is amenable.

Here amenable means $\|\Gamma\|^2 = [M:N]$ and $M \cong R$ by a theorem of Popa '95.

This result relies heavily on Popa '94 'Symmetric enveloping algebras, amenability and AFD properties of subfactors'.

In particular, if Γ is finite depth then $\sigma(\Phi({}_N L^2(M)_N)) = \sigma(\Gamma\Gamma^*)$ which only consists of algebraic integers.

If $\sigma(\Phi({}_N L^2(M)_N))$ contains non-algebraic integers then Γ is infinite depth.

Examples ① Let $q=p^m\equiv 1 \bmod 4$ be a prime power, $\mathbb{F}_q$ the Galois field and χ the quadratic character.

Then $H = \begin{bmatrix} 0 & 1 \cdots 1 \\ \vdots & \\ 1 & \chi(a-b)\end{bmatrix} \otimes \begin{bmatrix}1&1\\1&-1\end{bmatrix} + id_{q+1}\otimes\begin{bmatrix}1&-1\\-1&-1\end{bmatrix}$

is a Hadamard matrix and produces an infinite depth subfactor since

$$\frac{4^2 q}{(q+1)^2}\in\sigma\left(\Phi\left({}_N L^2(M)_N\right)\right).$$

② Petrescu's 7×7 continuous family of complex Hadamards all yield infinite depth subfactors.

③ Since the spectrum of a continuous family of s.a. matrices varies continuous infinite depth subfactors are typical among continuous families of biunitaries