COUNTABLE VS UNCOUNTABLE

BRENT NELSON

In class we showed that $\sqrt{2}$ is an *irrational number*. Hence $\mathbb{R} \setminus \mathbb{Q}$ is non-empty. In this note, we will see that, in fact, $\mathbb{R} \setminus \mathbb{Q}$ is quite large. Larger even than $\mathbb{Q}$. We will also make use of the decimal expansions we derived in class.

1. Countable Sets

On Homework 2, where it was shown $\mathbb{R} \setminus \mathbb{Q}$ is dense, we already observed that $\mathbb{R} \setminus \mathbb{Q}$ is an infinite set. Indeed, $Q + \sqrt{2} \subset \mathbb{R} \setminus \mathbb{Q}$. This is because if $x \in Q$, then $x + \sqrt{2} = y \in \mathbb{Q}$ would imply that $\sqrt{2} = y - x \in \mathbb{Q}$, a contradiction. Thus $\mathbb{R} \setminus \mathbb{Q}$ is at least as big is $\mathbb{Q} + \sqrt{2}$, and so in particular is infinite. However, we claim that $\mathbb{R} \setminus \mathbb{Q}$ is even larger. In order to compare different sizes of infinity, we will need the notion of *countability*.

Definition 1.1. A set S is said to be **countable** if there exists an onto function $f\colon \mathbb{N} \to S$.

The way you should think of this property is that countable sets can be written as (potentially infinite) lists. Indeed, if $f\colon \mathbb{N} \to S$ is onto, then the list $\{f(1), f(2), \ldots\}$ exhausts all of S. This implies that the natural numbers $\mathbb{N}$ (the "smallest" infinite set) are at least as big as S.

Example 1.2. Any finite set $S = \{s_1, \ldots, s_N\}$, $N \in \mathbb{N}$, is countable. Take the function

$$f(n) := \begin{cases} s_n & \text{if } n \leq N \\ s_N & \text{if } n > N \end{cases}$$

Example 1.3. The natural numbers $\mathbb{N}$ are countable. Take $f(n) := n$.

Example 1.4. The integers $\mathbb{Z}$ are countable. Take the function

$$f(n) := \begin{cases} k & \text{if } n = 2k \text{ for } k \in \mathbb{N} \\ -k & \text{if } n = 2k+1 \text{ for } k \in \mathbb{N} \cup \{0\} \end{cases}$$

Example 1.5. The rational numbers $\mathbb{Q}$ are countable. Take the function

$$f(n) := \begin{cases} \frac{a}{b} & \text{if } n = 2^a 3^b \text{ for } a, b \in \mathbb{N} \\ -\frac{a}{b} & \text{if } n = 2^a 3^b 5 \text{ for } a, b \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

The function in Example 1.4 is one-to-one as well as onto. This implies that $\mathbb{N}$ and $\mathbb{Z}$ are the same "size." The function in Example 1.5 is very much not one-to-one (it sends infinitely many numbers to zero). However, it is possible to come up with a function which is both one-to-one and onto, in which case $\mathbb{N}$ and $\mathbb{Q}$ are the same size.

Proposition 1.6. *Let I be a countable set, and for every $i \in I$ let S_i be a countable set. Then the union*

$$S = \bigcup_{i \in I} S_i$$

is countable.

Proof. Let $g\colon \mathbb{N} \to I$ be an onto function, and for each $i \in I$ let $f_i\colon \mathbb{N} \to S_i$ be an onto function. Define

$$h(n) := \begin{cases} f_{g(a)}(b) & \text{if } n = 2^a 3^b \text{ for } a, b \in \mathbb{N} \\ f_{g(1)}(1) & \text{otherwise} \end{cases}$$

For fixed $a \in \mathbb{N}$, $\{h(2^a 3^1), h(2^a 3^2), \ldots\} = \{f_{g(a)}(1), f_{g(a)}(2), \ldots\}$ exhausts $S_{g(a)}$. Then varying a exhausts I and covers the union S. □

What this proposition tells us, is that even increasing the size of a countable set by a countably infinite factor still yields a countable set. Thus *uncountable sets* must be truly enormous.

2. Uncountable Sets

Definition 2.1. A set S is **uncountable** if it is *not* countable.

This definition is unsurprising, but uncountable sets form a broad class and are further categorized by cardinal numbers, which can be thought of as different tiers of infinity.

The proof of the following theorem is known as Cantor's famous *diagonalization argument.*

Theorem 2.2. *$\mathbb{R}$ is uncountable.*

Proof. Suppose, towards a contradiction, that $\mathbb{R}$ is countable and hence that there exists an onto function $f\colon \mathbb{N} \to \mathbb{R}$. For each $n \in \mathbb{N}$, let

$$f(n) = a_0^{(n)}.a_1^{(n)}a_2^{(n)}\cdots$$

be the decimal expansion of $f(n)$, as defined in class. Recall that, by our procedure, this decimal expansion will never conclude with an infinite sequence of 9's.

Now, f being onto means that the list $\{f(1), f(2), \ldots\}$ should exhaust $\mathbb{R}$, but we will construct $x \in \mathbb{R}$ which is not on the list. For each $n \in \mathbb{N}$ define

$$b_n := \begin{cases} 2 & \text{if } a_n^{(n)} = 1 \\ 1 & \text{otherwise} \end{cases}$$

We then let

$$x = 0.b_1 b_2 \cdots = \sup\{0.b_1 b_2 \cdots b_n \colon n \in \mathbb{N}\} \in \mathbb{R}.$$

Now, for each $n \in \mathbb{N}$ the 10^{-n}-digits of x and $f(n)$ are b_n and $a_n^{(n)}$, respectively. But b_n was defined precisely so that $b_n \neq a_n^{(n)}$. Hence $x \neq f(n)$. Since this is true for every $n \in \mathbb{N}$, f cannot be onto, a contradiction. Thus $\mathbb{R}$ is uncountable. □

Corollary 2.3. *$\mathbb{R} \setminus \mathbb{Q}$ is a uncountable*

Proof. Since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$, if $\mathbb{R} \setminus \mathbb{Q}$ were countable then $\mathbb{R}$ would be countable by Proposition 1.6. By the previous theorem, we see that this cannot be the case. Hence $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. □

One can make use of this diagonalization argument to show other sets are uncountable. For example, the interval $[0, 1]$ or more generally any interval with non-empty interior is uncountable. Another example is the Cantor set, which is connected to many paradoxes. In particular, it is an example of an uncountable set but with "measure zero".