MONOTONE SEQUENCE OF CONTINUOUS FUNCTIONS

We give a sequential proof to Exercise 41 in Chapter IV of Rosenlicht.

Let (E, d) be a compact metric space and let $f_n \colon E \to \mathbb{R}$ be a continuous function for each $n \in \mathbb{N}$. Suppose $(f_n)_{n\in\mathbb{N}}$ converges pointwise to some continuous function $f \colon E \to \mathbb{R}$ and that

$$f_1(x) \le f_2(x) \le \cdots \qquad \forall x \in E.$$

We will show that, in fact, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

We first note that it suffices to show that $(f - f_n)_{n\in\mathbb{N}}$ converges uniformly to the zero function. Denote $h_n := f - f_n$ for each $n \in \mathbb{N}$. From the properties of the f_n, we know that $h_n(x) \ge 0$ for all $x \in E$, $(h_n)_{n\in\mathbb{N}}$ converges pointwise to zero, and that

$$h_1(x) \ge h_2(x) \ge \cdots \qquad \forall x \in E.$$

Since h_n is a continuous function on a compact metric space, it attains its maximum value. Let $x_n \in E$ be such that

$$h_n(x_n) = \max_{x\in E} h_n(x).$$

Note that for each $x \in E$,

$$|h_n(x) - 0| = h_n(x) \le h_n(x_n).$$

Thus if we can show $\lim\limits_{n\to\infty} h_n(x_n) = 0$, then we have shown $(h_n)_{n\in\mathbb{N}}$ converges uniformly to zero. Observe that

$$h_{n+1}(x_{n+1}) \le h_n(x_{n+1}) \le h_n(x_n),$$

so that the sequence $(h_n(x_n))_{n\in\mathbb{N}}$ is a monotone decreasing sequence of real numbers that is bounded below by zero. Consequently it converges, say to

$$y := \lim_{n\to\infty} h_n(x_n),$$

and $h_n(x_n) \ge y$ for all $n \in \mathbb{N}$. Suppose, towards a contradiction, that $y > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is a sequence in a compact metric space, it has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$, say with limit $x \in E$. Note that we still have

$$\lim_{k\to\infty} h_{n_k}(x_{n_k}) = y.$$

Now, since $(h_n)_{n\in\mathbb{N}}$ converges pointwise to zero, $(h_{n_k})_{k\in\mathbb{N}}$ converges pointwise to zero and in particular $\lim\limits_{k\to\infty} h_{n_k}(x) = 0$. Let $K \in \mathbb{N}$ be such that for all $k \ge K$ we have

$$|h_{n_k}(x) - 0| < \frac{y}{2}.$$

Equivalently, $h_{n_k}(x) < \frac{y}{2}$. Now, h_{n_K} is continuous at x, so there exists $\delta > 0$ such that if $x' \in E$ satisfies $d(x', x) < \delta$ then $|h_{n_K}(x') - h_{n_K}(x)| < \frac{y}{2}$. Consequently,

$$h_{n_K}(x') \le h_{n_K}(x') - h_{n_K}(x) + h_{n_K}(x) < \frac{y}{2} + \frac{y}{2} = y.$$

So by the monotonicity condition on the h_n, whenever $k \ge K$ we have

$$h_{n_k}(x') < y$$

so long as $d(x', x) < \delta$. We know that for sufficiently large k, x_{n_k} satisfies this. Taking $k \ge K$ as well we have

$$h_{n_k}(x_{n_k}) < y.$$

But since $y \le h_{n_k}(x_{n_k})$, this implies $y < y$, a contradiction.

Thus it must be that $y = 0$, which, as noted above, implies $(h_n)_{n\in\mathbb{N}}$ converges uniformly to zero. $\square$