A SEQUENCE WITH NO LIMIT

In the metric space $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$, we consider the following sequence:

$$x_n := \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases} .$$

We claim that this sequence has no limit. Let $x \in \mathbb{R}$. We must show that there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $d(x_n, x) \geq \epsilon$. We claim that $\epsilon = \frac{1}{2}$ works. Let $N \in \mathbb{N}$. We will produce our $n \geq N$ in two ways depending on how x compares to $\frac{1}{2}$.

Case 1: $x \leq \frac{1}{2}$. In this case, we have $d(1, x) \geq \frac{1}{2}$. Thus we let n be the first odd integer larger than N, so that $x_n = 1$ and we have $d(x_n, x) = d(1, x) \geq \frac{1}{2}$.

Case 2: $x > \frac{1}{2}$. Then $r := x - \frac{1}{2} > 0$. We let n be an even integer satisfying $n \geq \max\{N, \frac{1}{r}$. Then $\frac{1}{n} \leq r$ and $-\frac{1}{n} \geq -r$. Thus

$$d(x_n, x) = \left| \frac{1}{n} - x \right| = x - \frac{1}{n} \geq x - r = \frac{1}{2}.$$

Thus $\{x_n\}_{n \in \mathbb{N}}$ does not converge to x. Since $x \in \mathbb{R}$ was arbitrary, this sequence has no limit.