$$D(f_n, f_m) < \varepsilon$$

$\iff \sup\{d'(f_n(x), f_m(x)) : x \in E\} < \varepsilon$

$\implies \forall x \in E \quad d'(f_n(x), f_m(x)) < \varepsilon.$

So by an earlier prop, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some $f : E \to E'$. Since each $f_n$ is cts, the unif. conv. $\Rightarrow$ $f$ is cts and therefore $f \in C(E,E')$. Finally, as we saw above, $(f_n)_{n\in\mathbb{N}}$ conv unif. to $f$ $\iff$ $(f_n)_{n\in\mathbb{N}}$ conv to $f$ w.r.t $D$. Thus the Cauchy seq. $(f_n)_{n\in\mathbb{N}}$ conv. to $f$, and so $(C(E,E'), D)$ is complete. ∎

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~~Thus~~ We summarize the above as the following thm:

<u>Thm</u> Let $(E,d)$ be a compact metric space, and let $(E',d')$ be a complete metric space. If

$$C(E,E') = \{ f : E \to E' \mid f \text{ cts} \}$$

then

$$D(f,g) = \sup\{ d'(f(x), g(x)) : x \in E\} \qquad f,g \in C(E,E')$$

defines a metric on $C(E,E')$ s.t. $(C(E,E'), D)$ is complete. Moreover, $(f_n)_{n\in\mathbb{N}} \subseteq C(E,E')$ conv. ~~conv.~~ to $f \in C(E,E')$ w.r.t. $D$ iff $(f_n)_{n\in\mathbb{N}}$ conv unif. to $f$ on $E$.

## Differentiation V

- We ~~turn~~ focus now on functions

$$f : U \to \mathbb{R} \quad \text{with} \quad U \subseteq \mathbb{R} \text{ open.}$$

We wish to formalize the notion of a derivative. This will require limits of functions.

which you should recall relied on cluster points.
Fortunately, ~~every~~ this condition is mild in $\mathbb{R}$:

<u>Lemma</u>: For $U \subseteq \mathbb{R}$ open, every $x \in U$ is a cluster point of $U$.

<u>Pf</u>: Since $U$ is open, if $x \in U$ then $\exists r_0 > 0$ s.t. $B(x, r_0) \subseteq U$. Now, $x$ is a cluster point of $U$ iff $\forall r > 0$

$$B(x, r) \cap U \text{ is infinite}$$

But

$$\underset{\cup\!\!\mid}{B(x,r)} \quad B(x, \min\{r, r_0\}) = (x - \min\{r, r_0\}, x + \min\{r, r_0\})$$

and this interval is clearly infinite. $\square$

<u>Def</u>: Let $f: U \to \mathbb{R}$ with $U \subseteq \mathbb{R}$ open. For $x_0 \in U$, we say $f$ is differentiable at $x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. If it exists, ~~the~~ call this limit the derivative of $f$ at $x_0$ and denote it by $f'(x_0)$.
Equivalently,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if it exists.

• Recall the formal meaning of $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

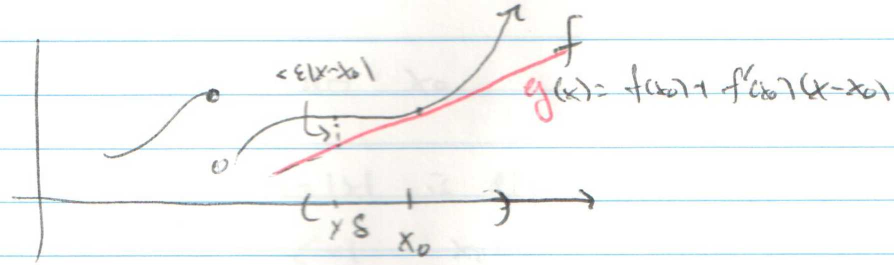
For $x_0 \in U$ $f'(x_0)$ is a number s.t.

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $|x - x_0| < \delta$ then $\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$

$$\Longleftrightarrow \Big| f(x) - \underbrace{[f(x_0) + f'(x_0)(x - x_0)]}_{g(x)} \Big| < \varepsilon |x - x_0|$$

$g(x)$ is a line ~~xxxxx~~

Thus, $f$ being diff'ble at $x_0$ means that for $x$ near $x_0$ (locally), $f(x)$ can be approximated by a linear function $g(x)$ up to a fraction of $|x - x_0|$.




**Prop** Let $U \subseteq \mathbb{R}$ be open and suppose $f: U \to \mathbb{R}$ is diff'ble at $x_0 \in U$. Then $f$ is cts at $x_0$.

**Pf**: Let $\delta_1 > 0$ be s.t. if $|x-x_0| < \delta_1$ then:

$$|f(x) - [f(x_0) + f'(x_0)\cdot(x-x_0)]| < 1\,|x-x_0|$$

Observe that if $|x-x_0| < \delta_1$ we have

$$|f(x) - f(x_0)| \le |f(x) - f(x_0) - f'(x_0)(x-x_0) + f'(x_0)\cdot(x-x_0)|$$
$$\le |f(x) - f(x_0) - f'(x_0)(x-x_0)| + |f'(x_0)(x-x_0)|$$
$$< |x-x_0| + |f'(x_0)|\cdot|x-x_0|$$
$$= |x-x_0|\,(1 + |f'(x_0)|).$$

Now, let $\varepsilon > 0$. Set $\delta = \min\{\delta_1, \frac{\varepsilon}{1+|f'(x_0)|}\}$.
Then if $|x-x_0| < \delta$, the above shows

$$|f(x) - f(x_0)| < |x-x_0|\cdot(1+|f'(x_0)|) < \delta(1+|f'(x_0)|) \le \varepsilon.$$

Thus $f$ is cts at $x_0$ $\square$

**Ex** (1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is diff'ble at every $x_0 \in \mathbb{R}$ with $f'(x) = 2x_0$.
Indeed, we "know" this to be the case from calculus. Fix $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$.
We'll determine $\delta > 0$ later: we estimate:

$$\left|\frac{f(x) - f(x_0)}{x-x_0} - 2x_0\right| = \left|\frac{x^2 - x_0^2}{x-x_0} - 2x_0\right| = |x + x_0 - 2x_0|$$
$$= |x - x_0|.$$

Thus if $\delta = \varepsilon$ and $|x-x_0| < \delta$ we have
$$< \varepsilon.$$

Thus $f$ is diff'ble at $x_0$

(2) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=|x|$ is diff'ble at $x_0\in\mathbb{R}\setminus\{0\}$
with

$$f'(x_0)=\begin{cases}-1 & x_0<0\\ 1 & 0<x_0\end{cases}$$

but not diff'ble at $0$.

First, fix $x_0\in\mathbb{R}\setminus\{0\}$, and let $\varepsilon>0$.

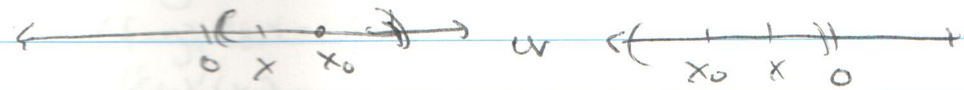
~~We'll need some care to estimate~~

$$\cancel{\left|\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)\right|}$$

~~we need to know if $x_0<0$ or $x_0>0$ (and if $x<0$ or $x>0$. If we know~~

We ~~haven't~~ set $\delta$ quite yet, but we will at least assume $\delta\le|x_0|$. Then, if $|x-x_0|<\delta$ we have:

$$\cancel{|x|=|x-x_0+x_0|}\ge|x_0|-|x-x_0|\ge|x_0|-\delta\ge 0.$$




That is, $x$ and $x_0$ have the same sign. When $x_0>0$ and if $|x-x_0|<\delta$, then $x>0$.
Thus

$$\left|\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)\right|=\left|\frac{x-x_0}{x-x_0}-1\right|=0<\varepsilon$$

Thus we only needed $\delta\le|x_0|$.

~~Now, for $x_0=0$, it's not hard to see that~~ $\cancel{\frac{f(x)-f(x_0)}{x-x_0}\ \begin{cases}-1\\1\end{cases}}$

~~Exercise: Show~~ $\cancel{\lim_{x\to x_0} f(x)-f(x_0)}$

Observe:

$$g(x):=\frac{f(x)-f(0)}{x-0}=\frac{|x|-0}{x-0}=\begin{cases}-1 & x<0\\ 1 & x>0\end{cases}$$

Exercise, show $\lim_{x\to 0} g(x)$ does not exist.

return to prop.

**Def**: Let $f: U \to \mathbb{R}$ w/ $U \subseteq \mathbb{R}$ open. If $f'(x_0)$ exists for all $x_0 \in U$, we say $f$ is differentiable (on $U$). Consequently, $x_0 \mapsto f'(x_0)$ defines a function

$$f': U \to \mathbb{R}$$

we call $f'$ the derivative of $f$. Also denote $f' = \frac{d}{dx} f = \frac{df}{dx}$

## ~~Ex~~ Rules of Differentiation V.2

**Prop**: Let $f, g: U \to \mathbb{R}$ for $U \subseteq \mathbb{R}$ open. If $f$ and $g$ are diff'ble at $x_0 \in U$, then so are $f \pm g$, $f \cdot g$, and, if $g(x_0) \neq 0$, $f/g$. Their derivatives are given by:

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$$
$$(f \cdot g)'(x_0) = f(x_0) g'(x_0) + f'(x_0) g(x_0)$$
$$(f/g)'(x_0) = \frac{g(x_0) f'(x_0) - f(x_0) g'(x_0)}{g(x_0)^2}$$

**Pf**: Simply use limit arithmetic.

$$(f \cdot g)'(x_0) = \lim_{x \to x_0} \frac{f(x) g(x) - f(x_0) g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) g(x) - f(x) g(x_0) + f(x) g(x_0) - f(x_0) g(x_0)}{x - x_0}$$

$$= \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}\right) + \left(g(x_0) \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right)$$

using cty of $f$ at $x_0$ $= f(x_0) \cdot g'(x_0) + g(x_0) f'(x_0)$.

Check rest at home. $\square$

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**Cor**: (i) If $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}$ open, diff'ble at $x_0 \in U$ then $\forall a \in \mathbb{R}$ $(a \cdot f)'(x_0) = a f'(x_0)$

(ii) $\forall n \in \mathbb{N}$, $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$.