Then

$$R(f, G, S) = \sum_{e} f(s_e) \cdot |R_e|$$

is a Riemann sum for f, where $|R_e|$ is the product of ~~the~~ its edge lengths (which we think of as its volume).

We define Riemann integrability in the same way as in $n=2$ case, and all the properties of the Riemann integral hold here as well. In particular, ① the Riemann + Lebesgue Thm holds, where a zero set $Z \subseteq \mathbb{R}^n$ is s.t. $\forall \varepsilon > 0$ Z can be covered by countably many open boxes R_e satisfying

$$\sum_e |R_e| < \varepsilon.$$

② Fubini's theorem also holds: by induction:

$$\int_R f = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \cdots dx_1$$

and this integral can be computed in any order

③ We can also prove the Change of Variables formula here, ~~(the volume~~ where $\mathrm{Jac}_z(\varphi)$ is the determinate of $(D\varphi)_z \in M(n,n)$. Moreover, the volume multiplier formula also holds, we just need to consider more elementary matrices.

5.8 Differential Forms

Recall Stokes theorem from Calculus: $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ $\vec{F}$ let $S \subseteq \mathbb{R}^3$ be a smooth surface with simple closed bounding curve C. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}.$$

You also learned Green's theorem and the divergence theorem, which are essentially consequences of Stokes' theorem derived by changing the dimension you're integrating over. We will make these rigorous corollaries by establishing a general form of Stokes' theorem in Section 5.9. First, we need to develop the theory of differential forms, which formalizes the familiar notation $f\,dx$.

Idea: Recall the path integral:

$$\int_C f\,dx + g\,dy = \int_0^1 f(x(t),y(t)) \cdot \frac{dx(t)}{dt}dt + \int_0^1 g(x(t),y(t)) \cdot \frac{dy(t)}{dt}dt$$

where $f, g: \mathbb{R}^2 \to \mathbb{R}$ are smooth and $C \subseteq \mathbb{R}^2$ is a smooth path parametrized by:

$$C = \{(x(t), y(t)) : 0 \le t \le 1\}$$

Normally, you think of this number as depending on f and g with C fixed. However, if we fix f and g, we can think of it as a function on smooth paths C.

Def A differential 1-form is a function that sends smooth paths to real numbers and which can be expressed as a path integral.

For f and g as above, we let $f\,dx + g\,dy$ denote the differential 1-form given by the above path integral

Ex ① With $f = 1$ and $g = 0$ $f\,dx + g\,dy =: dx$ we have

$$dx(C) = \int_C 1\,dx = \int_0^1 \frac{dx(t)}{dt}dt = x(1) - x(0)$$

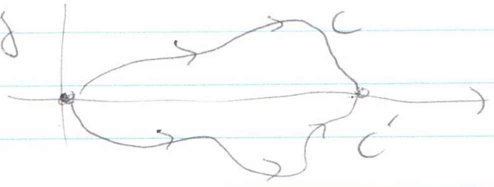
2/28/2016 So dy sends C to the net change in x-coordinate

~~Note that if we reverse the orientation of~~

our parametrization, we obtain a different curve C' with $dx(C') = -dx(C)$.
Thus forms are sensitive to orientation

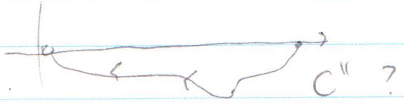
② $f\,dx$ is similar to dx but it weights the path by the value of C. If C passes through a region where f is large $f\,dx(C)$ will weight the variation in x accordingly.

E.g: $f = y$



$$f\,dx(C) > 0 > f\,dx(C').$$

what about:



Remark If C is curve and C' is curve w/ reverse orient, $(f\,dx + g\,dy)(C') = -(f\,dx + g\,dy)(C)$

Non-Ex ③ Let $\omega(C)$ be the arclength of C

$$\omega(C) = \int_0^1 \sqrt{\frac{dx(t)}{dt}^2 + \frac{dy(t)}{dt}} \, dt$$

This is not a differential 1-form. ~~Proof~~ since $\omega(C) = \omega(C')$ Indeed,

Suppose $\omega = f\,dx + g\,dy$ for some f, g.

~~Consider C: $x(t) = t$, $y(t) = 0$, $0 \le t \le 1$ →~~

~~Then~~

~~$1 = (f\,dx + g\,dy)(C) = f\,dx(C) = \int_0^1 f(t)\,dt$.~~

~~However, for C': $x(t) = 1 - t$, $y(t) = 0$, $0 \le t \le 1$~~

~~we have~~

~~$1 = (f\,dx + g\,dy)(C') = f\,dx(C') = -\int_0^1 f(t)\,dt$~~

~~contradiction.~~

Def: A functional on set X is a function from X to $\mathbb{R}$.

- Differential 1-forms are ~~all~~ functionals on smooth paths, but not all functionals are diff. 1-forms

We will later define "differential k-forms" for $k \in \mathbb{N}$, which will be certain functionals on the k-dimensional analogue of smooth paths. So we first formalize the domain of these functionals.

Def: For $k \in \mathbb{N}$, a k-cell in $\mathbb{R}^n$ is a *smooth map $\varphi : [0,1]^k \longrightarrow \mathbb{R}^n$. We call $[0,1]^k = I^k$ the unit k-cube. The set of k-cells in $\mathbb{R}^n$ is denoted $C_k(\mathbb{R}^n)$. The 1-cells in $\mathbb{R}^n$ are called paths. * really φ defined on $U \supset I^k$ open, φ smooth on U

- So we have considered 1-cells on $\mathbb{R}^2$, but we can of course consider $\varphi = (\varphi_1, \ldots, \varphi_n)$ a 1-cell in $\mathbb{R}^n$. In this case, a differential 1-form looks like $f_1 dx_1 + \ldots + f_n dx_n$ for smooth real-valued functions $f_1, \ldots, f_n$ and

$$(f_1 dx_1 + \ldots + f_n dx_n)(\varphi) = \int_0^1 f_1(\varphi(t)) \cdot \frac{d\varphi_1(t)}{dt} dt + \ldots + \int_0^1 f_n(\varphi(t)) \cdot \frac{d\varphi_n(t)}{dt} dt$$

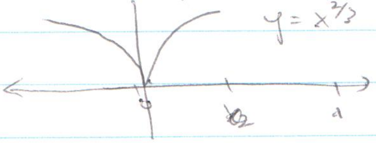
the usual path integral. Note that we are distinguishing between φ and $\varphi([0,1])$, since φ contains information about the orientation of $\varphi([0,1])$.

Remark. Note that we are not requiring φ be a diffeomorphism. In particular, φ may be non-injective and hence have $(D\varphi)_z = 0$.
While this allows cusps to appear in $C_1(\mathbb{R}^n)$, it also means the closed unit disc is a 2-cell in $\mathbb{R}^2$.

① consider $\varphi \in C_1(\mathbb{R}^2)$ defined by:

$$\varphi(t) = ((2t-1)^3, (2t-1)^2) \qquad 0 \le t \le 1$$

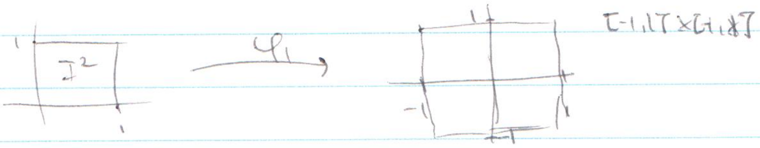
Then φ is clearly a smooth function (its entries are polynomials and hence smooth) However, $\varphi([0,1])$ is part of the graph of $y = x^{2/3}$.



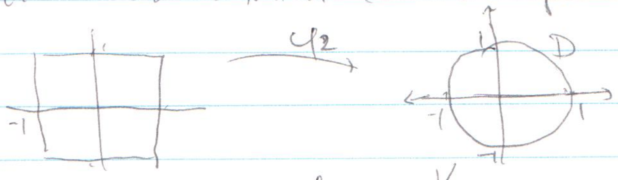
which has a cusp at $(0,0) = \varphi(0)$. This corresponds to

$$(D\varphi)_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

② $\exists\, \varphi \in C_2(\mathbb{R}^2)$ s.t. $\varphi(I^2) = D$ the closed unit ball in $\mathbb{R}^2$. Indeed, let $\varphi = \varphi_2 \circ \varphi_1$, where

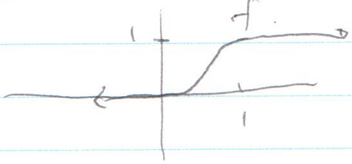


by translation and dilation (smooth operations) and



is given by $\varphi_2(v) = f(|v|) \cdot \frac{v}{|v|}$ where $f : \mathbb{R} \to [0,1]$ is any smooth function satisfying

$$\left.\begin{array}{ll} f(x) = 0 & \text{if } x \le 0 \\ f(x) = 1 & \text{if } x \ge 1 \end{array}\right\}$$



(e.g. use $\exp(-1/t^2)$ to produce f)

φ_2 and hence φ is not injective since

$$\varphi_2(av) = \frac{v}{|v|} \qquad \forall a \ge 1,$$

but φ is still smooth.

If we required that φ be a diffeomorphism, we could never have $\varphi(I^2) = D$ because of the corners in I^2.

Def: For $I \overset{=(i_1,\ldots,i_k)}{\in} \{1,\ldots,n\}^k$ and $\varphi \in C_k(\mathbb{R}^n)$, the Jacobian of φ_I at $u \in I^k$ is:

$$\frac{\partial \varphi_I(u)}{\partial u} = \det \begin{bmatrix} \frac{\partial \varphi_{i_1}(u)}{\partial u_1} & \cdots & \frac{\partial \varphi_{i_1}(u)}{\partial u_k} \\ \vdots & & \vdots \\ \frac{\partial \varphi_{i_k}(u)}{\partial u_1} & \cdots & \frac{\partial \varphi_{i_k}(u)}{\partial u_k} \end{bmatrix}$$

We may also write

$$\frac{\partial(\varphi_{i_1},\ldots,\varphi_{i_k})}{\partial(u_1,\ldots,u_k)} := \frac{\partial \varphi_I}{\partial u}$$

If $k=1$ so that $I=(i)$, we just write $\frac{\partial \varphi_i}{\partial u} := \frac{\partial \varphi_I}{\partial u}$.

Remark: The matrix whose determinant gives $\partial \varphi_I^{(u)}/\partial u$ is simply a square-submatrix of the possibly rectangular $(D\varphi)_u$.

Remark: If $i_a = i_b$ for any $a \neq b$, the determinant is zero since two distinct rows are identical. Consequently, in this case $\frac{\partial \varphi_I}{\partial u} = 0$.
If $k > n$, this is unavoidable, in which case $\frac{\partial \varphi_I}{\partial u} = 0$ for all $I \in \{1,\ldots,n\}^k$.
If $k = n$, and $I = (1,\ldots,n)$, then

$$\frac{\partial \varphi_I}{\partial u}(u) = \mathrm{Jac}_u(\varphi)$$

from the change of variables formula.

Def. For $I \in \{1,\ldots,n\}^k$ with distinct entries and $\varphi \in C_k(\mathbb{R}^n)$, the I-shadow area of φ is

$$dy_I(\varphi) := \int_{I^k} \frac{\partial \varphi_I}{\partial u}$$

(we use the notation 'y' to recognize that φ_I has had its range variables or coordinate functions restricted).

Remark: We should think of $dy_I(\varphi)$ as the "signed" area/volume of the projection of $\varphi(I^k)$ onto the $(y_{i_1}, \dots, y_{i_k})$-coordinate plane in $\mathbb{R}^n$.
Indeed, if $\psi: I^k \to \mathbb{R}^k$ is defined by

$$\psi(u) = (\varphi_{i_1}(u), \dots, \varphi_{i_k}(u))$$

then $\psi(I^k)$ is this projection of $\varphi(I^k)$.
If $\frac{\partial \varphi_I}{\partial u} > 0$ then

$$\frac{\partial \varphi_I(u)}{\partial u} = \mathrm{Jac}_u(\psi) = |\mathrm{Jac}_u(\psi)|$$

and hence by the change of variables formula:

$$dy_I(\varphi) = \int_{I^k} \frac{\partial \varphi_I}{\partial u} = \int_{I^k} |\mathrm{Jac}_u(\psi)| = \int_{I^k} 1 \circ \psi \cdot |\mathrm{Jac}_u(\psi)|$$
$$= \int_{\psi(I^k)} 1 = |\psi(I^k)|.$$

3/2/2016 If $\frac{\partial \varphi_I}{\partial u} \not> 0$, then we obtained a "signed" area.

Ex: ① Let $k=1$, $n=2$ so that $C_1(\mathbb{R}^2)$ are smooth paths in $\mathbb{R}^2$. Consider

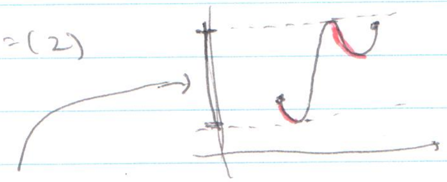
φ going from (x_0, y_0) to (x_1, y_1)
$C = \varphi([0,1])$

Then for $I = (1)$, we have:

$$dy_1(\varphi) = \int_0^1 \det\left[\frac{\partial \varphi_1}{\partial u_1}(u)\right] = \varphi_1(1) - \varphi_0(0)$$
$$= x_1 - x_0$$

the net x-variation of C.

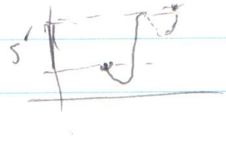
For $I = (2)$



- negative contribution

S projection of C onto y_2-coordinate plane but

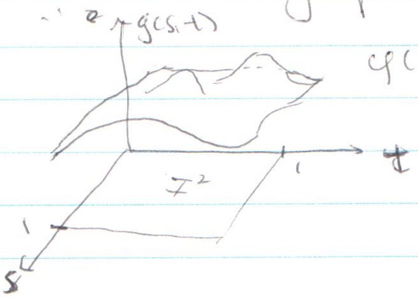
$$dy_2(\varphi) = y_1 - y_0 \neq |S|$$
$$= |S'| \text{ when}$$



② Let $g: I^2 \to \mathbb{R}$ be a smooth function and define $\varphi \in C_2(\mathbb{R}^3)$ by:

$$\varphi(s,t) = (s, t, g(s,t)) \qquad 0 \leq s,t \leq 1.$$

That is, $\varphi(I^2)$ is the graph of g:



(a) Let $I = (1,2)$, then

$$\frac{\partial \varphi_I}{\partial u}(u) = \det \begin{bmatrix} \frac{\partial \varphi_1}{\partial u_1} & \frac{\partial \varphi_1}{\partial u_2} \\ \frac{\partial \varphi_2}{\partial u_1} & \frac{\partial \varphi_2}{\partial u_2} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

Hence

$$dy_I(\varphi) = \int_{I^2} 1 = |I^2| = \text{proj. of } \varphi(I^2) \text{ onto } (y_1, y_2)\text{-plane}$$

(b) Let $I = (1,3)$, then

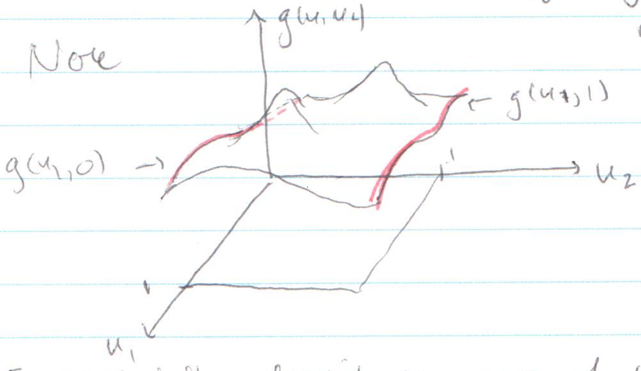
$$\frac{\partial \varphi_I}{\partial u} = \det \begin{bmatrix} 1 & 0 \\ \frac{\partial g}{\partial u_1}(u) & \frac{\partial g}{\partial u_2}(u) \end{bmatrix} = \frac{\partial g}{\partial u_2}$$

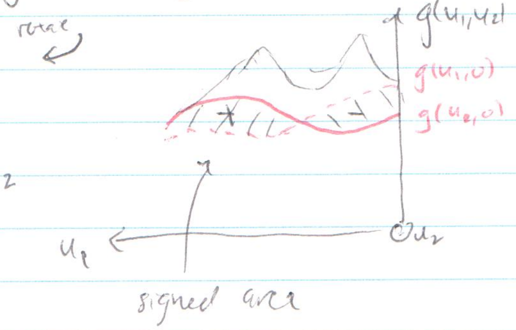
Thus

$$dy_I(\varphi) = \int_{I^2} \frac{\partial g}{\partial u_2} = \int_0^1 \int_0^1 \frac{\partial g}{\partial u_2}(u_1,u_2)\, du_2\, du_1$$

$$= \int_0^1 g(u_1,1) - g(u_1,0)\, du_1$$

Note





Exercise: Why doesn't the rest of the graph matter?

Observe that $\varphi \mapsto dy_I(\varphi)$ is a functional on the set of k-cells in $\mathbb{R}^n$.
Given $f: \mathbb{R}^n \to \mathbb{R}$ smooth we define a functional $f\,dy_I$ on $C_k(\mathbb{R}^n)$ by:

$$f\,dg_I(\varphi) := \int_{I^k} f(\varphi(u)) \cdot \frac{\partial \varphi_I}{\partial u}\,du$$

Remark: If $\boxed{k=n}$, (so that $\varphi_I = \varphi$), and $\frac{\partial \varphi_I}{\partial u} > 0$ then the change of variables formula says:

$$f\,dy_I(\varphi) = \int_{\varphi(I^k)} f$$

So in general, we think of f as weighting the I-shadow area.

Def: The functional dy_I on k-cells is called a basic (differential) k-form and $f\,dy_I$ is called a simple (differential) k-form.
If $I_1, \ldots, I_m \in \{1, \ldots, n\}^k$ and $f_1, \ldots, f_m: \mathbb{R}^n \to \mathbb{R}$ are smooth, then

$$\sum_{j=1}^{n} f_j\, d_{I_j}$$

is called a general (differential) k-form

Notation: Given a $\overset{\text{general}}{}$ k-form ω and $\varphi \in C_k(\mathbb{R}^n)$ we write

$$\int_\varphi \omega := \omega(\varphi)$$

The set of all functionals on $C_k(\mathbb{R}^n)$ will be denoted $C^k(\mathbb{R}^n)$, while the set of general k-forms on $\mathbb{R}^n$ will be denote $\Omega^k(\mathbb{R}^n)$. So $\Omega^k(\mathbb{R}^n) \subsetneq C^k(\mathbb{R}^n)$.

Remark: Since the sign of the determinant changes under row transposition: if π permutes I to πI, then

$$f\,dy_{\pi I} = \mathrm{sgn}(\pi) f\,dy_I$$

where $\mathrm{sgn}(\pi)$ is the number of transpositions in π. This property is called signed commutativity

Remark: As with ~~[illegible]~~ case $\frac{\partial y_I}{\partial u}$, $f\,dy_I(\varphi) = 0$ $\forall \varphi \in C_k(\mathbb{R}^n)$ if I has any repeated entries. ~~E.g. [illegible]~~

3/5/2018

Form Naturality

~~Def~~: $T: U \to W$ diffeo is orientation preserving/reversing if $\frac{\partial T}{\partial u} > 0 / \frac{\partial T}{\partial u} < 0$

Thm: Let $T: I^k \to I^k$ be a diffeomorphism (i.e. a reparameterization of I^k). Then for any $\varphi \in C_k(\mathbb{R}^n)$ and $\omega \in \Omega^k(\mathbb{R}^n)$ we have $\varphi \circ T \in C_k(\mathbb{R}^n)$ with

$$\int_{\varphi \circ T} \omega = \pm \int_\varphi \omega$$

where '$\pm$' is determined by whether T is orientation preserving (+), or not (−).

Pf: It suffices to prove this for $\omega = f\,dy_I$ a simple k-form. Since T is a diffeo, its Jacobian determinant

$$\mathrm{Jac}_u(T) = \frac{\partial T(u)}{\partial u} \in \mathbb{R}$$

is cts and non-zero. Hence it is either always positive or always negative. The former corresponds to when T is orientation preserving (in fact, this is a definition of orientation preserving), and the latter to orientation reversing.

Assume the former. Then

$$\int_{\varphi \circ T} \omega = \int_{I^k} f(\varphi \circ T(u)) \frac{\partial (\varphi \circ T)_I(u)}{\partial u}\, du$$

$$\textcircled{=} \int_{I^k} f(\varphi \circ T(u)) \left(\frac{\partial \varphi_I}{\partial v}\right)_{(T(u))} \cdot \frac{\partial T(u)}{\partial u}\, du$$

$\frac{\partial T}{\partial u} = Jac_u(T) = |Jac_u(T)|$

Now, ~~by~~ the change of variables formula

$$\overset{\circ}{=} \int_{T(I^k)} f(\varphi(v)) \frac{\partial \varphi_I(v)}{\partial v} dv$$

$$= \int_{I^k} f(\varphi(v)) \frac{\partial \varphi_I(v)}{\partial v} dv = \int_{\varphi} \omega$$

If T is orientation reversing, we pick up the negative sign from

$$\frac{\partial I(u)}{\partial u} = -|Jac_u(T)|$$

$\square$

Remark: This theorem says, as far as k-forms are concerned, the only difference between φ and $\varphi(I^k)$ is (~~[illegible]~~ orientation. In particular, parametrization doesn't matter. This called form naturality.

~~Remark: The same proof shows that for $k=1$, $n=2$, if~~

~~$\varphi = \{(x(t), y(t)) : 0 \le t \le 1\}$~~

~~is reparameterized by arclength $s \in [0, L]$~~

Form Names

Def: we say $A = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ ascends if $i_1 < i_2 < \cdots < i_k$

Prop: Every $\omega \in \Omega^k(\mathbb{R}^n)$ as a unique expression as a sum of simple-k-forms indexed by ascending k-tuples:

$$\omega = \sum_A f_A \, dy_A \leftarrow \text{ascending presentation}$$

Moreover, ~~[illegible]~~ for an ascending k-tuple A and $y \in \mathbb{R}^n$ $f_A(y)$ is determined by ~~[illegible]~~ the values of ω on small cubes ~~centered at~~ containing y in the y_A-coordinate plane.

Pf: For $\omega \in \Omega^k(\mathbb{R}^n)$, we have

$$\omega = \sum f_I \, dy_I$$

without repeating entries (otherwise $dy_I = 0$)

for some collection of k-tuples I. For each I, $\exists!$ permutation π s.t. $A := \pi I$ ascends. Then

$$dy_A = \operatorname{sgn}(\pi)\, dy_I.$$

So define $f_A = \frac{1}{\operatorname{sgn}(\pi)} f_I$. Then

$$\omega = \sum f_A \, dy_A$$

Thus the ascending presentation exists. Uniqueness will follow by our characterization of $f_A(y)$.

Fix an ascending k-tuple $A = (i_1, \ldots, i_k)$ and fix $y \in \mathbb{R}^n$. Let $L_A : \mathbb{R}^k \to \mathbb{R}^n$ be defined by:

$$L_A(u_1, \ldots, u_k) = u_1 e_{i_1} + \cdots + u_k e_{i_k}$$

For $r > 0$, define $i_{r,y} : I^k \to \mathbb{R}^n$ by

$$i_{r,y}(u) = y + r L_A(u)$$

So $i_{r,y}(I^k)$ is a cube in the y_A-plane with sidelength r and containing $y (= i_{r,y}(0, \ldots, 0))$. Note that as $r \to 0$, this cube shrinks to y.

If $I \in \{1, \ldots, n\}^k$ is ascending, then

$$\frac{\partial (i_{r,y})_I}{\partial u} = \begin{cases} r^k & \text{if } I = A \\ 0 & \text{otherwise} \end{cases} \Bigg\} \text{exercise}$$

Hence, $f_I \, dy_I(i_{r,y}) = 0$ if $I \neq A$ and so:

$$\omega(i_{r,y}) = f_A \, dy_A(i_{r,y}) = r^k \int_{I^k} f_A(i_{r,y}(u))\, du$$

Since f_A is cts, we have:

$$\lim_{r \to 0} \frac{1}{r^k} \omega(i_{r,y}) = \lim_{r \to 0} \int_{I^k} f_A(i_{r,y}(u))\, du = f_A(y).$$

Hence f_A is unique. $\square$

The following corollary also follows from our earlier observation that $\frac{\partial f_I}{\partial u} = 0$ ~~when~~ when $k > n$

<u>Cor</u>: If $k > n$, $\Omega^k(\mathbb{R}^n) = 0$

<u>Pf</u>: There are no ascending k-tuples in $\{1, \dots, n\}$. $\square$

Thus we can uniquely identify $\omega \in \Omega^k(A)$ by the coefficients f_A in its ascending presentation.

Wedge Products

We define product structure on $\bigcup_{k=1}^{n} \Omega^k(\mathbb{R}^n)$ (by the above Cor we don't allow $k > n$).

<u>Def</u>: Let $\alpha \in \Omega^k(\mathbb{R}^n)$, $\beta \in \Omega^l(\mathbb{R}^n)$ with ascending presentations

$$\alpha = \sum a_A \, dy_A \qquad \beta = \sum b_B \, dy_B .$$

Their wedge ~~presentation~~ product, denoted $\alpha \wedge \beta$, is the $(k+l)$-form

$$\alpha \wedge \beta = \sum_{A,B} a_A \, b_B \, dy_{AB}$$

where the sum is over ascending

$$A = (i_1, \dots, i_k)$$
$$B = (j_1, \dots, j_l)$$

and

$$AB = (i_1, \dots, i_k, j_1, \dots, j_l)$$

(which is not necessarily ascending).

– Point: use ascending presentations to ensure $\wedge$ is well-defined.

3/7/2018 Ex: ① $(f \, dy_i) \wedge (g \, dy_j) = f \cdot g \, dy_{(i,j)}$

② $(f \, dy_I) \wedge (g \, dy_J) = f g \, dy_{IJ}$ for I, J ascending

where $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$

Remark: Since $dy_I = 0$ whenever I repeat indices, for $dy_I \in \Omega^k(\mathbb{R}^n)$, $dy_J \in \Omega^\ell(\mathbb{R}^n)$, $dy_I \wedge dy_J = 0$ if $k+\ell > 0$.
and $dy_I \wedge dy_I = 0$.

(63)

③ $dy_2 \wedge dy_1 = dy_{(2,1)} = -dy_{(1,2)} = (-1)\, dy_1 \wedge dy_2$.

④ $dy_1 \wedge dy_1 = dy_{(1,1)} = 0$

⑤ $dy_{(2,1)} \wedge dy_3 \overset{?}{=} dy_{(2,1,3)}$

Thm: The wedge product $\wedge : \Omega^k \times \Omega^\ell \to \Omega^{k+\ell}$ satisfies:

(a) distributivity: $(\alpha+\beta)\wedge\gamma = \alpha\wedge\gamma + \beta\wedge\gamma$ and
$$\gamma\wedge(\alpha+\beta) = \gamma\wedge\alpha + \gamma\wedge\beta$$

(b) insensitivity to presentations: $\alpha\wedge\beta = \sum_{I,J} a_I b_J\, dy_{IJ}$ for any presentations:
$$\alpha = \sum a_I\, dy_I \quad \text{and} \quad \beta = \sum b_J\, dy_J$$

(c) associativity: $\alpha\wedge(\beta\wedge\gamma) = (\alpha\wedge\beta)\wedge\gamma$

(d) signed commutativity: $\beta\wedge\alpha = (-1)^{k\ell}\alpha\wedge\beta$ for $\alpha\in\Omega^k$, $\beta\in\Omega^\ell$. In particular $dx\wedge dy = -dy\wedge dx$.

Lemma For arbitrary tuples I and J (not-necessarily ascending)
$$dy_I \wedge dy_J = dy_{IJ}$$

Pf: $\exists$ permutations π and ρ s.t. πI and ρJ are ascending. Then
$$dy_I \wedge dy_J = \operatorname{sgn}(\pi)\cdot\operatorname{sgn}(\rho)\, dy_{\pi I}\wedge dy_{\rho J}$$
$$= \operatorname{sgn}(\pi)\cdot\operatorname{sgn}(\rho)\, dy_{(\pi I)(\rho J)}$$

Let σ be the permutation that is π on the first $|I|$ terms and ρ and the last $|J|$ terms. Then
$$(\pi I)(\rho J) = \sigma(IJ).$$

Furthermore,
$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\pi)\cdot\operatorname{sgn}(\rho)$$
so continuing our previous computation, we obtain:
$$dy_I \wedge dy_J = \operatorname{sgn}(\sigma)\, dy_{\sigma(IJ)} = dy_{IJ}. \quad \square$$

Proof of Theorem:

(a) write in ascending presentations:
$$\alpha = \sum a_I\, dy_I \in \Omega^k(\mathbb{R}^n) \quad \beta = \sum \beta_I\, dy_I \in \Omega^k(\mathbb{R}^n)$$

and

$$\gamma = \sum c_J \, dy_J \in \Omega^\ell(\mathbb{R}^n)$$

Then $\alpha+\beta$ has ascending presentation

$$\alpha+\beta = \sum (a_I + b_I)\, dy_I$$

~~[illegible] ascending presentation~~ so by the def. ~~wedge~~ of the wedge product we have:

$$(\alpha+\beta)\wedge\gamma = \sum (a_I+b_I)\, c_J \, dy_{IJ}$$
$$= \sum a_I c_J \, dy_{IJ} + \sum b_I c_J \, dy_{IJ}$$
$$= \alpha\wedge\gamma + \beta\wedge\gamma.$$

The proof of $\gamma \wedge (\alpha+\beta)$ is similar.

(b) Let

$$\alpha = \sum a_I \, dy_I \in \Omega^k(\mathbb{R}^n) \qquad \beta = \sum b_J \, dy_J \in \Omega^\ell(\mathbb{R}^n)$$

be general presentations. Using distributivity and the lemma, we obtain:

$$\alpha\wedge\beta = \sum_I \left(a_I \, dy_I \wedge \left(\sum b_J \, dy_J\right)\right)$$
$$= \sum_I \sum_J (a_I \, dy_I) \wedge (b_J \, dy_J)$$
$$= \sum_{I,J} a_I b_J \, dy_{IJ}$$

(c) By part (b), we need not worry about using ascending presentations anymore. Let

$$\alpha = \sum a_I \, dy_I$$
$$\beta = \sum b_J \, dy_J$$
$$\gamma = \sum c_K \, dy_K$$

Then

$$\alpha\wedge(\beta\wedge\gamma) = \alpha \wedge \left(\sum_{J,K} b_J c_K \, dy_{JK}\right) = \sum_{I,J,K} a_I b_J c_K \, dy_{IJK}$$

and

$$(\alpha\wedge\beta)\wedge\gamma = \left(\sum_{I,J} a_I b_J \, dy_{IJ}\right)\wedge\gamma = \sum_{I,J,K} a_I b_J c_K \, dy_{IJK}$$

which agree.

(d) By part (c), there is no ambiguity if for $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_\ell)$ we write

$dy_{i_1}\wedge\cdots\wedge dy_{i_k}$ and $dy_{j_1}\wedge\cdots\wedge dy_{j_\ell}$
for dy_I and dy_J, respectively. Thus

$$dy_I\wedge dy_J = dy_{i_1}\wedge\cdots\wedge dy_{i_k}\wedge dy_{j_1}\wedge\cdots\wedge dy_{j_\ell}$$

We know

$$dy_i\wedge dy_j = (-1)\, dy_j\wedge dy_i$$

So in $dy_I\wedge dy_J$, it takes $(-1)^k$ to move each $dy_{j_1},\dots,dy_{j_\ell}$ past $dy_{i_1},\dots,dy_{i_k}$.
So in total we have

$$dy_I\wedge dy_J = (-1)^{k\ell}\cdot dy_{j_1}\wedge\cdots\wedge dy_{j_\ell}\wedge dy_{i_1}\wedge\cdots\wedge dy_{i_k}$$

$$= (-1)^{k\ell}\, dy_J\wedge dy_I.$$

This implies the formula for general $\alpha\in\Omega^k(\mathbb{R}^n)$ and $\beta = \Omega^\ell(\mathbb{R}^n)$ using distributivity. □

The Exterior Derivative

We will define a derivative on forms in such a way that the derivative of k-forms gives (k+1)-forms. To motivate this, let's talk about "0-forms". Formally, it should be certain kind of functional on "0-cells" which are smooth functions from I^0 (a singleton set) into $\mathbb{R}^n$. That is, a 0-cell in $\mathbb{R}^n$ is just a point. But then a functional on a point is simply a function: $f:\mathbb{R}^n\to\mathbb{R}$. Hence a 0-form on $\mathbb{R}^n$ is a smooth function

$$f:\mathbb{R}^n\longrightarrow\mathbb{R}$$

Now we show define a functional df on 1-cells in $\mathbb{R}^n$ by:

$$df(\varphi) = f(\varphi(1)) - f(\varphi(0)) \qquad \varphi\in C_1(\mathbb{R}^n)$$

i.e. the net f-variation along φ

Prop For $f: \mathbb{R}^n \to \mathbb{R}$ a smooth function, df is the 1-form defined by:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

Pf: ~~[illegible]~~

write $\omega = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$

For $\varphi \in C_1(\mathbb{R}^n)$ we have:

$$\int_\varphi \omega = \int_0^1 \frac{\partial f}{\partial x_1}(\varphi(t)) \frac{d\varphi_1(t)}{dt} dt + \dots + \frac{\partial f}{\partial x_n}(\varphi(t)) \frac{d\varphi_n(t)}{dt} dt$$

$$= \int_0^1 \frac{d}{dt}(f \circ \varphi)(t)\, dt$$

$$= f \circ \varphi(1) - f \circ \varphi(0)$$

by the chain-rule and the fundamental theorem of calculus. □

• Observe that for $(x,y) \mapsto x$, $d(x) = dx$

$(x,y) \mapsto y$, $d(y) = dy$.

○ That is, the net x and y variation are the image of these smooth functions under d, ~~where for arbitrary smooth $f: \mathbb{R}^n \to \mathbb{R}$, df is the net f variation.~~

5/9/2017

Def: Fix $k \geq 1$, let $\omega = \Sigma f_A dy_A \in \Omega^k(\mathbb{R}^n)$ be in ascending presentation. The exterior derivative of ω is

$$d\omega := \sum_A df_A \wedge dy_A \in \Omega^{k+1}(\mathbb{R}^n)$$

Remark: $d\omega$ really depends on how each f_A changes. If f_A is a constant, $df_A = 0$ since the net f_A-variation will always be zero. In particular, we always have

$$d(dy_I) = 0.$$

We define the exterior derivative on ascending presentations to ensure it is well defined. But just like with wedge products, the presentation will not affect computations.

Ex: Let $\omega = f\,dx + g\,dy \in \Omega^1(\mathbb{R}^2)$. Then by def. of the exterior deriv:

$$d\omega = df \wedge dx + dg \wedge dy.$$

Using the earlier proposition we have

$$d\omega = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)\wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)\wedge dy$$

$$= 0 + \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial g}{\partial x}dx \wedge dy + 0$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx \wedge dy$$

Observe that ω and $d\omega$ are precisely the integrands on the two-sides of Green's Thm:

$$\int_C f\,dx + g\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy.$$

Thm Exterior differentiation $d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ satisfies:

(a) Linearity: $d(\alpha + c\beta) = d\alpha + c\,d\beta$

(b) Insensitivity to presentation: for general presentations $\omega = \sum f_I\,dy_I$

$$d\omega = \sum_I df_I \wedge dy_I$$

(c) Product Rule: for $\alpha \in \Omega^k$, $\beta \in \Omega^\ell$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

(d) $d^2 = 0$; that is, $d(d\omega) = 0$ $\forall \omega \in \Omega^k$.

Proof:

(a) Clear.

(b) For $I \in \{1, \dots, n\}^k$, let π be a permutation sending

I to ascending πI. Then the linearity of d and the associativity of $\wedge$ implies

$$d(f_I\, dy_I) = d(\operatorname{sgn}(\pi)\, f_I\, dy_{\pi I})$$

$$= \operatorname{sgn}(\pi)\, d(f_I \wedge dy_{\pi I})$$

$$= \operatorname{sgn}(\pi)\, df_I \wedge dg_{\pi}I$$

$$= df_I \wedge (\operatorname{sgn}(\pi) \cdot dy_{\pi I})$$

$$= df_I \wedge dy_I.$$

Linearity then yields the formula for $\Sigma f_I\, dy_I$.

(c) We first note that the Leibniz rule for partial derivatives implies for $f, g : \mathbb{R}^n \to \mathbb{R}$

$$d(fg) = \frac{\partial(fg)}{\partial x_1} dx_1 + \cdots + \frac{\partial(fg)}{\partial x_n} dx_n$$

$$= \left(\frac{\partial f}{\partial x_1} \cdot g + f \cdot \frac{\partial g}{\partial x_1}\right) dx_1 + \cdots + \left(\frac{\partial f}{\partial x_n} \cdot g + f \cdot \frac{\partial g}{\partial x_n}\right) dx_n$$

$$= g \cdot \left(\frac{\partial f}{\partial x_1} \cdot dx_1 + \cdots + \frac{\partial f}{\partial x_n} \cdot dx_n\right) + f\left(\frac{\partial g}{\partial x_1} dx_1 + \cdots + \frac{\partial g}{\partial x_n} \cdot dx_n\right)$$

$$= \cancel{g(df \wedge f\, dg)}\ g\, df + f \cdot dg$$

Thus

$$d(f\, dy_I \wedge g\, dy_J) = d(fg\, dy_{IJ})$$

$$= d(fg) \wedge dy_{IJ} = (g\, df + f\, dg) \wedge dy_{IJ}$$

$$= (df \wedge dy_I) \wedge (g\, dy_J) + (-1)^{k \cdot 1} (f\, dy_I) \wedge (dg \wedge dy_J)$$

$$= d(f\, dy_I) \wedge (g\, dy_J) + (-1)^k (f\, dy_I) \wedge d(g\, dy_J)$$

So the product rule holds on simple forms. Using the distributivity of $\wedge$ and linearity of d yields the product rule for general forms.

(d) We previously noted that for a basic k-form

$$d(dy_I) = d(1) \wedge dy_I = 0.$$

Consider $f : \mathbb{R}^n \to \mathbb{R}$ smooth. Then

$$
\begin{aligned}
d^2 f &= d\left(\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \right) \\
&= d\left(\frac{\partial f}{\partial x_1}\right) \wedge dx_1 + \cdots + d\left(\frac{\partial f}{\partial x_n}\right) \wedge dx_n \\
&= \left(\frac{\partial^2 f}{\partial x_1 \partial x_1} dx_1 + \cdots + \frac{\partial^2 f}{\partial x_n \partial x_1} dx_n \right) \wedge dx_1 + \cdots + \left(\frac{\partial^2 f}{\partial x_1 \partial x_n} dx_1 + \cdots + \frac{\partial^2 f}{\partial x_n^2} dx_n \right) \wedge dx_n \\
&= \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} \underbrace{dx_j \wedge dx_j}_{=0} + \sum_{1 \le i < j \le n} \underbrace{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right)}_{=0} dx_i \wedge dx_j \\
&= 0.
\end{aligned}
$$

Thus for a simple k-form $f\,dy_I$ we have by Product rule:

$$d^2(f\,dy_I) = d(df \wedge dy_I) = (d^2 f) \wedge dy_I + (-1)^1 df \wedge d(dy_I)$$

$$= 0 + 0.$$

Linearity then implies $d^2 \omega = 0$ $\forall \omega \in \Omega^k(\mathbb{R}^n)$. $\square$

Remark: Thinking of $f: \mathbb{R}^n \to \mathbb{R}$ smooth as an element of $\Omega^0(\mathbb{R}^n)$, and defining

$$f \wedge dy_I := f\,dy_I,$$

then the ~~previous~~ definition of the exterior derivative is actually using the product rule and $d^2 = 0$:

$$d(f\,dy) = d(f \wedge dy) = df \wedge dy_I + (-1)^0 f \wedge d(dy_I)$$

$$= df \wedge dy_I + 0.$$

3/12/2018 ~~Push forward~~

Pushforward and Pullback

Form naturality showed how k-cells and the action of k-forms on k-cells was affected by reparameterization: a factor of ±1. This studied changes to the domain of a q-cell, but what happens if we change

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Have range space?

Def Fix a smooth function

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

Then for $\varphi \in C_k(\mathbb{R}^n)$, the pushforward of φ by T is the k-cell in $\mathbb{R}^m$, $T_*\varphi \in C_k(\mathbb{R}^m)$ defined

$$T_*\varphi = T\circ\varphi$$

(which is clearly a smooth map $I^k \to \mathbb{R}^m$)

Now, for any k-form on $\mathbb{R}^n$ $\alpha \in \Omega^k(\mathbb{R}^m)$ we can consider

$$\int_{T_*\varphi}\alpha = \alpha(T_*\varphi) = \alpha(T\circ\varphi).$$

Thinking dually, we can view the above as a change to α, rather than a change to φ. That is

$$* \qquad \varphi \mapsto \alpha(T\circ\varphi)$$

defines a functional on $C_k(\mathbb{R}^n)$.

Def: The ~~pushforward~~ pullback of α by T is the functional on $C_k(\mathbb{R}^n)$ defined by $(*)$, and is denoted $T^*\alpha$. The relation

$$T^*\alpha(\varphi) = \alpha(T_*\varphi)$$

for $\alpha \in \Omega^k(\mathbb{R}^m)$ and $\varphi \in C_k(\mathbb{R}^n)$ is called the duality equation

We will show that $T^*\alpha$ is in fact a k-form on $\mathbb{R}^m$, so that

$$T^*: \Omega^k(\mathbb{R}^{\boxed{m}}) \longrightarrow \Omega^k(\mathbb{R}^{\textcircled{n}})$$

while

$$T_*: C_k(\mathbb{R}^{\textcircled{n}}) \longrightarrow C_k(\mathbb{R}^{\boxed{m}})$$

and recall

$$T: \mathbb{R}^{\textcircled{n}} \longrightarrow \mathbb{R}^{\boxed{m}}$$

- For $f \in \Omega^0(\mathbb{R}^m)$, $[T^*(f)](\overset{\overset{\mathbb{R}^n \ni C_0(\mathbb{R}^n)}{\downarrow}}{y}) = f(T_* y) = f(T(y)) = f \circ T(y)$.

  So $T^*(f) = f \circ T$.

- For $\alpha, \beta \in \Omega^k(\mathbb{R}^m)$, $\varphi \in C_k(\mathbb{R}^n)$, and $\lambda \in \mathbb{R}$ we have:

$$[T^*(\underbrace{\alpha+\lambda\beta}_{\Omega^k(\mathbb{R}^m)})](\varphi) = [\alpha+\lambda\beta](T_*\varphi) = [\alpha+\lambda\beta](T\circ\varphi)$$
$$= \alpha(T\circ\varphi) + \lambda\beta(T\circ\varphi) = T^*\alpha(\varphi) + \lambda T^*\beta(\varphi)$$

So $T^*(\alpha+\lambda\beta) = T^*\alpha + \lambda T^*\beta$; that is, T^* is linear on $\Omega^k(\mathbb{R}^m)$.

- __Lemma__ (Cauchy-Binet formula)

  For $k \le n$, $A \in M(k,n)$ and $B \in M(n,k)$

$$\det(AB) = \sum_I \det(A^I)\det(B_I)$$

~~where $A^I \in M(k,k)$ is the submatrix of A formed~~

where the sum is over ascending k-tuples $I = (i_1, \ldots, i_k)$, $A^I \in M(k,k)$ is the submatrix of A formed by columns $i_1, \ldots, i_k$ of A, and $B_I \in M(k,k)$ is the submatrix of B formed by rows $i_1, \ldots, i_k$ of B.

__Pf__ Exercise — read Appendix E.

- In addition to the above lemma, we will need the following (well-known) explicit formula for the determinant of $A \in M(k,k)$:

$$** \quad \det(A) = \sum_{\pi \in S_k} \operatorname{sgn}(\pi)\, [A]_{1,\pi(1)} \cdot [A]_{2,\pi(2)} \cdots [A]_{k,\pi(k)}$$

__Thm__ Pullbacks of forms obey the following five natural conditions:

(a) The pullback of a form is a form: for $f\,dz_I \in \Omega^k(\mathbb{R}^m)$ and $T: \mathbb{R}^n \to \mathbb{R}^m$

$$T^*(f\,dz_I) = \sum_A T^*(f)\, \frac{\partial T_I}{\partial y_A}\, dy_A \in \Omega^k(\mathbb{R}^n)$$

where the sum ranges over ascending $A = (a_1, \ldots, a_k) \in \{1, \ldots, n\}^k$ and

$$\frac{\partial T_I}{\partial y_A} = \frac{\partial(T_{i_1}, \ldots, T_{i_k})}{\partial(y_{a_1}, \ldots, y_{a_k})} = \det\begin{bmatrix} \frac{\partial T_{i_1}}{\partial y_{a_1}} & \cdots & \frac{\partial T_{i_1}}{\partial y_{a_k}} \\ \vdots & & \\ \frac{\partial T_{i_k}}{\partial y_{a_1}} & \cdots & \frac{\partial T_{i_k}}{\partial y_{a_k}} \end{bmatrix}$$

In particular

$$T^*(dz_I) = dT_I := dT_{i_1} \wedge \cdots \wedge dT_{i_k}$$

(b) The pullback preserves wedge products: $T^*(\alpha \wedge \beta) = (T^*\alpha) \wedge (T^*\beta)$

(c) The pullback commutes w/ the ext. derivative: $T^* d = d T^*$

(d) The pullback commutes w/ the integral: $\int_{T_*\varphi} \alpha = \int_\varphi T^*\alpha$.

Proof:

(a). Let $f\,dz_I \in \Omega^k(\mathbb{R}^m)$ and $\varphi \in C_k(\mathbb{R}^n)$.

We compute:

$$[T^*(f\,dz_I)](\varphi) = f\,dz_I(T \circ \varphi)$$

$$= \int_{I^k} f \circ T \circ \varphi \; \frac{\partial (T\circ\varphi)_I(u)}{\partial u} du$$

Now, $(T\circ\varphi)_I = T_I \circ \varphi$ and the Cauchy-Binet formula implies:

$$\frac{\partial (T_I \circ \varphi)(u)}{\partial u} = \sum_A \left(\frac{\partial T_I}{\partial y_A}\right)_{y=\varphi(u)} \cdot \left(\frac{\partial \varphi_A(u)}{\partial u}\right)$$

where A ranges over ascending k-tuples in $\{1, \ldots, n\}$.

So continuing our computation we have:

$$[T^*(f\,dz_I)](\varphi) = \sum_A \int_{I^k} (f\circ T)(\varphi(u)) \cdot \left(\frac{\partial T_I(y)}{\partial y_A}\right)_{y=\varphi(u)} \cdot \frac{(\partial \varphi_A)(u)}{\partial u} du.$$

On the other hand:

$$\left[\sum_A f \circ T \cdot \frac{\partial T_I(y)}{\partial y_A} dy_A\right](\varphi) = \sum_A \int_{I^k} (f\circ T)(\varphi(u)) \cdot \left(\frac{\partial T_I(y)}{\partial y_A}\right)_{y=\varphi(u)} \frac{\partial \varphi_A(u)}{\partial u} du$$

So

$$T^*(f\,dz_I) = \sum_A f \circ T \; \frac{\partial T_I(y)}{\partial y_A} dy_A.$$

as claimed

[Notation note: we could just write $\frac{\partial T_I(\varphi(u))}{\partial y_A}$ instead of $\left(\frac{\partial T_I(y)}{\partial y_A}\right)_{y=\varphi(u)}$, but the latter makes the role of y_A (which we are partially differentiating with respect to) clearer.]

The linearity of T^*, which we checked earlier, shows that

$$T^* : \Omega^k(\mathbb{R}^m) \longrightarrow \Omega^k(\mathbb{R}^n).$$

It remains to check the formula for $T^*(dz_I)$. Let $I = (i_1, \dots, i_k)$. Then by definition of the ext. deriv. on 0-forms and the distributivity of the wedge product we have:

$$dT_I = dT_{i_1} \wedge \dots \wedge dT_{i_k} = \left(\sum_{s_1=1}^{n} \frac{\partial T_{i_1}}{\partial y_{s_1}} dy_{s_1}\right) \wedge \dots \wedge \left(\sum_{s_k=1}^{n} \frac{\partial T_{i_k}}{\partial y_{s_k}} dy_{s_k}\right)$$

$$= \sum_{s_1, \dots, s_k=1}^{n} \frac{\partial T_{i_1}}{\partial y_{s_1}} \cdots \frac{\partial T_{i_k}}{\partial y_{s_k}} dy_{s_1} \wedge \dots \wedge dy_{s_k}.$$

Note that if the k-tuple $(s_1, \dots, s_k)$ has any repeating entries then $dy_{s_1} \wedge \dots \wedge dy_{s_k} = 0$. So we can reduce the above sum to only k-tuples $(s_1, \dots, s_k)$ w/ no repeated entries. In this case, each $(s_1, \dots, s_k) = \pi A$ for a unique ascending k-tuple A and unique permutation π.
So by signed commutativity:

$$dT_I = \sum_{A=(a_1, \dots, a_k)} \left(\sum_{\pi \in S_k} \mathrm{sgn}(\pi) \frac{\partial T_{i_1}}{\partial y_{\pi(a_1)}} \cdots \frac{\partial T_{i_k}}{\partial y_{\pi(a_k)}}\right) dy_A$$

$$= \sum_A \frac{\partial T_I}{\partial y_A} dy_A$$

where we have used (**) in the last equality.
By our previous work, this equals $T^*(dz_I)$.

(b) Let $f, g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions, thought of as elements of $\Omega^0(\mathbb{R}^m)$. Then

$$T^*(f \wedge g) = T^*(fg) = (fg) \circ T = (f \circ T)(g \circ T)$$
$$= (T^*f)(T^*g)$$

Now, let $\alpha = f\, dz_I \in \Omega^k(\mathbb{R}^m)$ and $\beta = g\, dz_J \in \Omega^l(\mathbb{R}^m)$.
Then by (a)

$$T^*(\alpha \wedge \beta) = T^*(fg\, dz_{IJ}) = (fg) \circ T\, dT_{IJ}$$
$$= (T^*f)(T^*g)\, dT_I \wedge dT_J$$
$$= (T^*\alpha) \wedge (T^*\beta)$$

Wedge distrib. and linearity of T^* completes the proof.

(c) For $f \in \Omega^0(\mathbb{R}^m)$ we compute

$$T^*(df) = T^*\left(\sum_{i=1}^{m} \frac{\partial f}{\partial z_i} dz_i\right)$$

(linearity) $$= \sum_{i=1}^{m} T^*\left(\frac{\partial f}{\partial z_i} dz_i\right)$$

(part (a)) $$= \sum_{i=1}^{m} \left(\frac{\partial f}{\partial z_i} \circ T\right) \cdot \sum_{j=1}^{n} \frac{\partial T_i}{\partial y_j} dy_j$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \left(\frac{\partial f}{\partial z_i}\right) \circ T \cdot \frac{\partial T_i}{\partial y_j} \right) dy_j$$

(chain rule) $$= \sum_{j=1}^{m} \frac{\partial (f \circ T)}{\partial y_j} dy_j$$

$$= d(f \circ T) = d(T^* f)$$

Thus $T^* d = dT^*$ on 0-forms.

Now, let $\alpha = f\, dz_I \in \Omega^k(\mathbb{R}^m)$. By part (a) we have

$$T^*(f\, dz_I) = \sum_A T^*(f) \frac{\partial T_I}{\partial y_A} dy_A$$

$$= T^*(f)\, dT_I$$

So

$$d(T^*(f\, dz_I)) = d(\underline{T^*(f)}\, dT_I)$$

$$= d(T^*(f)) \wedge dT_I$$

$$= T^*(df) \wedge dT_I$$

$$= T^*(df) \wedge T^*(dz_I)$$

(part (b)) $$= T^*(df \wedge dz_I)$$

$$= T^*(d(f\, dz_I)).$$

Linearity of T^* and d yields $T^* d = dT^*$ on all $\Omega^k(\mathbb{R}^m)$.

(d) This is just restating the duality equation:

$$\int_{T_*\varphi} \alpha = \alpha(T_*\varphi) = (T^*\alpha)(\varphi) = \int_\varphi T^*\alpha \qquad \square$$

Remark: As we saw in the proof of part (c), the two formulas from part (a) can be combined to the more easily stated

$$T^*(f\,dz_I) = T^*(f)\,dT_I.$$

Note also that since $f\,dz_I = f \wedge dz_I$ and $dT_I = T^*(dz_I)$, this is really just saying that the pullback preserves wedge products.

5.8 General Stokes Formula

We will prove the following formula:

$$\int_\varphi d\omega = \int_{\partial\varphi} \omega$$

for $\omega \in \Omega^k(\mathbb{R}^n)$ and $\varphi \in C_{k+1}(\mathbb{R}^n)$, where '$\partial\varphi$' will represent a "boundary" for the k-cell φ, that we will make formal. From this formula, we will derive the formulas you saw in multivariable calculus: Green's theorem, Divergence theorem, Stokes' theorem.

Def: A k-chain in $\mathbb{R}^n$ is a formal linear combination of k-cells in $\mathbb{R}^n$:

$$\Phi = \sum_{j=1}^{N} a_j \varphi_j$$

$a_1, \dots, a_N \in \mathbb{R}$, $\varphi_1, \dots, \varphi_N \in C_k(\mathbb{R}^n)$. The integral of $\omega \in \Omega^k(\mathbb{R}^n)$ over the k-chain Φ is defined as

$$\int_\Phi \omega := \sum_{j=1}^{N} a_j \int_{\varphi_j} \omega.$$

Remark: We really mean formal sum here, because the explicit sum (thinking of $\varphi_1, \dots, \varphi_N : [0,1]^k \to \mathbb{R}^n$)