6 Lebesgue Theory

6.1 Outer Measure

Our initial goal is to produce a rigorous way to measure the size of objects in $\mathbb{R}^n$. Riemann measurability is a good start, but as we've seen even some reasonable sets ($\mathbb{Q}^2 \subseteq \mathbb{R}^2$) fail to be Riemann measurable. To achieve a more robust ~~condition~~ notion of measurability, we will adapt our notion of zero ~~measure~~ set. First, we establish some terminology:

__Def__: Let $n \in \mathbb{N}$. An open box in $\mathbb{R}^n$ is a subset of the form:

$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$$

for $a_i < b_i$, $i = 1, \ldots, n$. We write $|B| = \prod_{i=1}^{n} (b_i - a_i) = \text{volume}(B)$

For $n=1$, open box = open interval, $|B| = \text{length}(B)$

For $n=2$, open box = open rectangle, $|B| = \text{area}(B)$

__Def__: For $A \subseteq \mathbb{R}^n$, the n-dimensional outer measure of A is the quantity:

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} |B_k| : \{B_k\}_{k \in \mathbb{N}} \text{ is a countable collection of open boxes s.t. } A \subset \bigcup_{k=1}^{\infty} B_k \right\}$$

Note that $m^*(A)$ could be $+\infty$ if every series $\sum |B_k|$ diverges.

$\boxed{\text{Exercise: Show } m^* \text{ is translation invariant: } m^*(A + v) = m^*(A),\ v \in \mathbb{R}^n}$

__Remark__: It is clear that for $n=2$, if A is a zero set, then $m^*(A) = 0$

Def: If $A \subseteq \mathbb{R}^n$ has $m^*(A) = 0$, we call A a zero set, or a Lebesgue zero set, or a null set.

Let's check that the outer measure gives us a reasonable answer on closed boxes:

Prop Let

$$A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

be a closed box. Then $m^*(A) = \prod_{i=1}^{n} (b_i - a_i)$

Pf: First observe that $\forall \varepsilon > 0$

$$B = (a_1 - \varepsilon, b_1 + \varepsilon) \times \cdots \times (a_n - \varepsilon, b_n + \varepsilon) \supseteq A.$$

Thus

$$m^*(A) \leq |B| = \prod_{i=1}^{n} (b_i - a_i + 2\varepsilon)$$

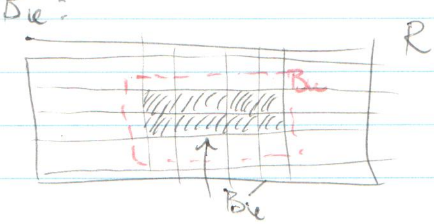
Letting $\varepsilon \to 0$ we have

$$m^*(A) \leq \prod_{i=1}^{n} (b_i - a_i)$$

To see the other inequality, fix a the covering of A by open boxes $\{B_k\}_{k \in \mathbb{N}}$. Since A is a closed, bdd set in $\mathbb{R}^n$ it is compact. Hence the open covering $\{B_k\}_{k \in \mathbb{N}}$ has a positive Lebesgue number $\lambda > 0$. Partition each $[a_i, b_i]$ into subintervals, so that R is partitioned into subboxes $\{S_{ij} : i \in I, j \in J\}$ of diameter $< \lambda$. Then each S_{ij} is completely contained in some B_k. Note that for fixed k,

$$B_k' := \bigcup_{S_{ij} \subseteq B_k} S_{ij}$$

is a subbox of B_k:



Hence $|B_k'| \leq |B_k|$. Thus we have:

$$\prod_{i=1}^{n}(b_i-a_i) = |A| = \sum_{\substack{i\in I\\ j\in J}} |S_{ij}| \leq \sum_{k\in 1}^{\infty} \sum_{S_{ij}\subseteq B_k} |S_{ij}|$$
$$= \sum_{k=1}^{\infty} |B_k'| \leq \sum_{k=1}^{\infty} |B_k|.$$

Since this holds for an arbitrary open covering $\{B_k\}_{k\in\mathbb{N}}$, we have
$$\prod_{i=1}^{n}(b_i-a_i) \leq m^*(A). \qquad \square$$

In particular, this ~~prop~~ implies
$$m^*([a,b]) = b-a$$
$$m^*([a,b]\times[c,d]) = (b-a)(d-c).$$
We should also have
$$m^*([a,b)) = m^*((a,b]) = m^*((a,b)) = b-a,$$
but to see this, we need the following theorem:

<u>Thm</u>: (a) $m^*(\phi)=0$

(b) If $A\subseteq B\subseteq \mathbb{R}^n$, then $m^*(A)\leq m^*(B)$ (<u>monotonicity</u>)

(c) If $A=\bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{R}^n$, then $m^*(A)\leq \sum_{i=1}^{\infty} m^*(A_i)$ (<u>countable subadditivity</u>)

<u>Pf</u>: (a) Since $\phi \subseteq (-\varepsilon,\varepsilon)^n = B$
$$0\leq m^*(\phi) \leq (2\varepsilon)^n \xrightarrow{\varepsilon\to 0} 0$$
Thus $m^*(\phi)=0$

(b) This follows from the fact that any open covering of B is an open covering of A.

(c) Let $\varepsilon>0$. For each $i\in\mathbb{N}$, let $\{B_k^{(i)}\}_{k\in\mathbb{N}}$ be an open covering of A_i s.t.
$$\sum_{k=1}^{\infty} |B_k^{(i)}| \leq m^*(A_i) + \frac{\varepsilon}{2^i}$$

Then $\{B_k^{(i)} : k\in\mathbb{N}, i\in\mathbb{N}\}$ is a c'ble open covering of A. Thus
$$m^*(A) \leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |B_k^{(i)}| \leq \sum_{i=1}^{\infty}\left(m^*(A_i)+\frac{\varepsilon}{2^i}\right)$$
$$=\left(\sum_{i=1}^{\infty} m^*(A_i)\right)+\varepsilon$$

Letting $\varepsilon\to 0$ completes the proof. $\square$

Cor Let
$$A = I_1 \times \cdots \times I_n \in \mathbb{R}^n$$
where each I_i is of the form
(a_i, b_i), $[a_i, b_i)$, $(a_i, b_i]$, or $[a_i, b_i]$.
Then
$$m^*(A) = \prod_{i=1}^{n} (b_i - a_i).$$

Pf: ~~Let~~ $\varepsilon > 0$ and consider
$$A' = [a_1+\varepsilon, b_1-\varepsilon] \times \cdots \times [a_n+\varepsilon, b_n-\varepsilon] \subseteq A$$
$$A'' = [a_1-\varepsilon, b_1+\varepsilon] \times \cdots \times [a_n-\varepsilon, b_n+\varepsilon] \supseteq A.$$
Then by the prop. and monotonicity we have:
$$\prod_{i=1}^{n} (b_i - a_i - 2\varepsilon) = m^*(A') \le m^*(A) \le m^*(A'') = \prod_{i=1}^{n} (b_i - a_i + 2\varepsilon).$$
Letting $\varepsilon \to 0$ yields the desired equality $\square$

Cor (a) Any subset of a zero set is a zero set.
(b) The countable union of zero sets is a zero set.
(c) Any countable set is a zero set
(d) Any plane $P_i(a) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = a\}$ is a zero set.

Pf: (a) This follows by monotonicity of m^*
(b) This follows by ctble subadditivity of m^*
(c) By the previous corollary
$$m^*(\{(x_1, \ldots, x_n)\}) = m^*([x_1, x_1] \times \cdots \times [x_n, x_n])$$
$$= \prod_{i=1}^{n} x_i - x_i = 0$$
So the outer measure of a single point is zero. Then any ~~ctbl~~ ctble set is therefore a zero set by part (b).
(d) Observe that

$$\cancel{\{((a_1, a_1+1) \times \cdots \times (a_{i-1}, a_{i-1}+1) \times (a-}$$
$$\{R_k = (-\tfrac{k}{2}, \tfrac{k}{2}) \times \cdots \times \underbrace{(a - \tfrac{\varepsilon}{2^k k^{n-1}}, a + \tfrac{\varepsilon}{2^k k^{n-1}})}_{i\text{th position}} \times \cdots \times (-\tfrac{k}{2}, \tfrac{k}{2}) : k \in \mathbb{N}\}$$

is a (the) open covering of $P_i(a)$ with
$$\sum_{k=1}^{\infty} |B_k| = \sum_{k=1}^{\infty} k^{n-1} \cdot \frac{2\varepsilon}{2^{k+1} k^{n-1}} = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$
Letting $\varepsilon \to 0$ yields $m^*(P_i(a)) = 0$. $\square$

6.2 Measurability

Our definition of the outer measure yields a map(s)
$$m^* : 2^{(\mathbb{R}^n)} \longrightarrow [0, +\infty].$$
which is certainly more robust than Riemann measurability. However, for mostly set theoretic reasons (and to avoid things like the Banach Tarski paradox) we will want to restrict m^* to a ~~subcollection~~ collection of subsets of $\mathbb{R}^n$.

Def: A set $E \subset \mathbb{R}^n$ is said to be Lebesgue measurable (or just measurable) if $\forall X \subseteq \mathbb{R}^n$ we have
$$* \quad m^*(X) = m^*(X \cap E) + m^*(X \cap E^c)$$
The set of Lebesgue measurable subsets is denoted $\mathcal{M}$ $(= \mathcal{M}(\mathbb{R}^n))$. For $E \subseteq \mathcal{M}$, the Lebesgue measure of E is the quantity:
$$m(E) := m^*(E)$$
(That is, the $m(E)$ is just the outer measure, but we drop the '$*$' only if E is measurable).
((*) is also called, more generally the Carathéodory condition)
The reason we use (*) as the criterion for measurability is the following: ~~[illegible]~~ lemma:
~~are disjoint, then~~
~~$m(A \cup B) = m(A) + m(B)$~~
~~Indeed, using $X = A$ in (*), we have~~