which, as we have seen, does not affect the integrals. $\square$

<u>Exercise</u>: ~~Apply~~ Check this corollary on (a) $\chi_{(n,n+1)}$ and (b) $n\chi_{(0,1/n)}$.

<u>Exercise</u>: Suppose $f \in L^+$ satisfies $\int f\,dm < \infty$. Show that $f < \infty$ a.e.

Integrating $\overline{\mathbb{R}}$-valued Functions

Suppose $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is Lebesgue measurable. Then if

$$E = \{x \in \mathbb{R}^d : f(x) \geq 0\} = f^{-1}([0,\infty]),$$

we have $E \in \mathcal{L}$ and so

$$f_+ := f\chi_E \quad \text{and} \quad f_- := -f\chi_{E^c}$$

are elements of L^+ with

$$f = f_+ - f_-.$$

We want to define the Lebesgue integral of f as

$$\int f_+\,dm - \int f_-\,dm,$$

but if $\int f_\pm\,dm = \infty$, we cannot make sense of "$\infty - \infty$". However, note that

$$|f| = f_+ + f_- \in L^+$$

If $\int |f|\,dm < \infty$, then $\int f_\pm\,dm < \infty$ and so we can reconcile $\int f_+\,dm - \int f_-\,dm$.

<u>Def</u>: We say a ~~[illegible]~~ Lebesgue measurable function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is ~~[illegible]~~ <u>Lebesgue integrable</u> if $\int |f|\,dm < \infty$.
(Equivalently, if $\int f_+\,dm, \int f_-\,dm < \infty$.)
In this case we define the <u>Lebesgue integral</u>

of f to be the quantity

$$\int f\,dm := \int f_+\,dm - \int f_-\,dm.$$

We denote by $L^1(\mathbb{R}^d, m)$ (or just $L^1(m)$) the set of Lebesgue integrable functions.

Exercise Show that $L^1(\mathbb{R}^d, m)$ is a vector space over $\mathbb{R}$, on which $f \mapsto \int f\,dm$ defines a linear functional.

Prop: If $f \in L^1(m)$, then $|\int f\,dm| \le \int |f|\,dm$

Pf:

$$\left|\int f\,dm\right| = \left|\int f_+\,dm - \int f_-\,dm\right| \le \int f_+\,dm + \int f_-\,dm$$
$$= \int |f|\,dm. \qquad \square$$

~~Prop: Let $f,g \in L^1(m)$, $\int_E f\,dm = \int_E g\,dm$ for all~~

Prop: Let $f, g \in L^1(m)$. TFAE:

(i) $\int_E f\,dm = \int_E g\,dm$ for all $E \in \mathcal{M}$.

(ii) $\int |f-g|\,dm = 0$

(iii) $f = g$ a.e.

Pf (i)$\Rightarrow$(ii): We have $\int |f-g|\,dm = \int (f-g)_+\,dm + \int (f-g)_-\,dm$.

Let

$$E_+ = \{x : f(x) \ge g(x)\}, \quad E_- = \{x : f(x) < g(x)\}$$

Then

$$(f-g)_\pm = (f-g)\chi_{E_\pm}$$

Thus

$$\int (f-g)_\pm\,dm = \int (f-g)\chi_{E_\pm}\,dm = \int_{E_\pm} (f-g)\,dm = 0.$$

So $\int |f-g|\,dm = 0$.

(ii)$\Leftrightarrow$(iii): $\int |f-g|\,dm \Leftrightarrow |f-g| = 0$ a.e. $\Leftrightarrow f = g$ a.e.

(iii)$\Rightarrow$(i) $f = g$ a.e. $\Rightarrow f\chi_E = g\chi_E$ a.e. $\forall E \in \mathcal{M} \Rightarrow$ (i). $\square$

Rem: This prop tells us that with respect to integration, modifying ~~on~~ functions on measure zero sets has no effect (as was so for measurability & measures of sets). Hence, since $\int |f| \, dm < \infty \Rightarrow f$ is finite a.e., we can treat every $f \in L^1(m)$ as finitely valued everywhere by setting it to, say, 0 wherever it was originally infinitely valued.

In fact, we can go further: we can redefine $L^1(m)$ as the set of equivalence classes of Lebesgue integrable functions under the equiv. relation:

$$f \sim g \Leftrightarrow f = g \text{ a.e.}$$

Advantage: In this case, for $f, g \in L^1(m)$ we have $\int |f-g| \, dm = 0$ iff $f = g$ (as equiv. classes)

Consequently, $L^1(m)$ is a metric space with metric:

$$d(f,g) := \int |f-g| \, dm$$

(Exercise: check other metric space properties)

Disadvantage: It no longer makes sense to discuss the value of $f \in L^1(m)$ at a point, since up to a.e. equiv, the value can be anything

(However, if one's goal to understand quantum anything* this de-emphasizing of the space $\mathbb{R}^d$ is a step in the right direction.)

Thm (Dominated Convergence Theorem)

Let $(f_n)_{n \in \mathbb{N}} \subseteq L^1(m)$ be a sequence s.t.

(a) $f_n \to f$ a.e.

(b) $\exists \, g \in L^1(m)$ s.t. $|f_n| \leq g$ a.e. for all $n \in \mathbb{N}$

Then $f \in L^1(m)$ with

$$\int f \, dm = \lim_{n \to \infty} \int f_n \, dm.$$

Exercise: Check why this theorem doesn't apply to $f_n = \chi_{(n,n+1)}$ or $f_n = n\chi_{(0,1/n)}$.

Proof (DCT): By an earlier prop, f is measurable, and $|f| \le |g|$ a.e. Hence

$$\int |f|\,dm \le \int |g|\,dm < \infty,$$

so $f \in L^1(m)$. Note that $g \pm f_n \ge 0$ a.e. So by Fatou's Lemma we have:

$$\int g\,dm \pm \int f\,dm = \int (g \pm f)\,dm \le \liminf_{n\to\infty} \int (g \pm f_n)\,dm$$
$$= \int g\,dm + \liminf_{n\to\infty} \int (\pm f_n)\,dm$$

So, subtracting $\int g\,dm$ (finite) from each side, we have:

$$\int f\,dm \le \liminf_{n\to\infty} \int f_n\,dm$$
$$-\int f\,dm \le -\limsup_{n\to\infty} \int f_n\,dm$$

or:

$$\int f\,dm \le \liminf_{n\to\infty} \int f_n\,dm \le \limsup_{n\to\infty} \int f_n\,dm \le \int f\,dm$$

So all the inequalities are equalities implying $\lim_{n\to\infty} \int f_n\,dm$ exists and equals $\int f\,dm$. $\square$

Thm Suppose $(f_n)_{n\in\mathbb{N}} \subseteq L^1(m)$ satisfies $\sum_{n=1}^{\infty} \int |f_n|\,dm < \infty$. Then $\sum_{n=1}^{\infty} f_n$ converges a.e. to a function in $L^1(m)$ and

$$\int \sum_{n=1}^{\infty} f_n\,dm = \sum_{n=1}^{\infty} \int f_n\,dm.$$

Pf: By the first theorem after the MCT

$$\int \sum_{n=1}^{\infty} |f_n|\,dm = \sum_{n=1}^{\infty} \int |f_n|\,dm < \infty.$$

So ~~we can apply the DCT with~~ $g = \sum_{n=1}^{\infty} |f_n| \in L^1(m)$ ~~but first~~, Recall that this implies g is finite a.e. and so the series converges a.e. Since

$$\left|\sum_{n=1}^{N} f_n\right| \le \sum_{n=1}^{N} |f_n| \le g \qquad \forall N \in \mathbb{N}$$

the DCT implies

$$\int \sum_{n=1}^{\infty} f_n \, dm = \int \lim_{N\to\infty} \sum_{n=1}^{N} f_n \, dm$$

$$(\text{DCT}) = \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n \, dm$$

$$(\text{linearity of int.}) = \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n \, dm$$

$$= \sum_{n=1}^{\infty} \int f_n \, dm. \qquad \square$$

<u>Thm</u>: If $f \in L^1(m)$, for any $\varepsilon > 0$ $\exists$ $\phi \in L^1(m)$ simple s.t.

$$\int |f - \phi| \, dm < \varepsilon.$$

<u>Pf</u>: Recall we proved $\exists$ $(\phi_k)_{k\in\mathbb{N}}$ simple s.t.

$$0 \le |\phi_1| \le |\phi_2| \le \dots \le |f|$$

s.t. $\phi_k \to f$ pointwise. Then $|f - \phi_k| \le 2|f| \in L^1(m)$ for all $k \in \mathbb{N}$ and $|f - \phi_k| \to 0$ pointwise.

So by the DCT

$$\lim_{k\to\infty} \int |f - \phi_k| \, dm = \int 0 \, dm = 0.$$

Thus $\forall \varepsilon > 0$, $\exists$ $k_0 \in \mathbb{N}$ s.t. $\phi = \phi_{k_0}$ satisfies

$$\int |f - \phi| \, dm < \varepsilon.$$

$\square$

<u>Rem</u>: This says that simple functions are dense in the metric space $(L^1(m), \int |\cdot - \cdot| \, dm)$.

<u>Thm</u>: If $f \in L^1(m)$, for any $\varepsilon > 0$, $\exists$ g cts and compactly supported s.t.

$$\int |f - g| \, dm < \varepsilon.$$

<u>Pf</u>: $\boxed{\text{Step 1}}$ Suppose $f = \chi_E$ for $E \in \mathcal{M}$ with

$$m(E) = \int \chi_E \, dm < \infty.$$

By cty from below,

$$m(E) = \lim_{n\to\infty} m(E \cap B(0,n))$$

So $\exists$ $n \in \mathbb{N}$ s.t. $m(E \cap B(0,n)) > m(E) - \frac{\varepsilon}{2}$.

Denote $E_0 := E \cap B(0,n)$. Observe that
$$\int |\chi_{E_0} - \chi_E|\, dm = \int \chi_{E \setminus E_0}\, dm$$
$$= m(E \setminus E_0) = m(E) - m(E_0) < m(E) - m(E) + \frac{\varepsilon}{2}$$
$$= \varepsilon/2.$$

Now, since $E_0 \in \mathcal{M}$ $\exists$ a G_δ set and an F_σ set
$$G = \bigcap_{i=1}^{\infty} U_i \leftarrow \text{open} \qquad F = \bigcup_{j=1}^{\infty} V_j \leftarrow \text{closed}$$
s.t. $m(G) = m(F) = m(E_0)$ and $G \supseteq E \supseteq F$. Since E_0 is bdd, our original construction showed each U_i is bdd. By cty from above and below resp.
$$m(E_0) = \lim_{I \to \infty} m\left(\bigcap_{i=1}^{I} U_i\right) \qquad m(E_0) = \lim_{J \to \infty} m\left(\bigcup_{j=1}^{J} V_j\right)$$

So $\exists\, I, J \in \mathbb{N}$ s.t.
$$m\left(\bigcap_{i=1}^{I} U_i\right) < m(E_0) + \frac{\varepsilon}{4}$$
and
$$m\left(\bigcup_{j=1}^{J} V_j\right) > m(E_0) - \frac{\varepsilon}{4}.$$
Set $U := \bigcap_{i=1}^{I} U_i$, $V = \bigcup_{j=1}^{J} V_j$. Then $\underset{\text{closed}}{V} \subseteq E_0 \subseteq \underset{\text{open, bdd}}{U}$ and
$$m(U \setminus V) = m(U) - m(V) < m(E_0) + \frac{\varepsilon}{4} - m(E_0) + \frac{\varepsilon}{4}$$
$$= \frac{\varepsilon}{2}.$$

[Note that $V \subseteq E_0 \subseteq B(0,n)$, so is compact as a closed and bdd set.] Define
$$g(x) = \frac{d(x, U^c)}{d(x, U^c) + d(x, V)},$$
where for a set S
$$d(x, S) = \inf\{d(x,y) : y \in S\}.$$ compact support

It follows that $0 \le g \le 1$, $g|_{U^c} = 0$, $g|_V = 1$, and (exercise) g is cts. Then
$$\int |g - \chi_{E_0}|\, dm \le \int \chi_{U \setminus V}\, dm = m(U \setminus V) < \frac{\varepsilon}{2}$$

Putting this together with our previous estimate, we have $\int |\chi_E - g|\,dm < \varepsilon$.

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$\boxed{\text{Step 2}}$ Let $f \in L^1(m)$ be arbitrary. By the previous theorem, $\exists\ \phi \in L^1(m)$ simple s.t.

$$\int |f - \phi|\,dm < \frac{\varepsilon}{2}.$$

Suppose $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$. Then, applying $\boxed{\text{Step 1}}$ for each j we find g_j cts with compact support s.t.

$$\int |g_j - \chi_{E_j}|\,dm < \frac{\varepsilon}{2N|a_j|}.$$

Then $g = \sum_{j=1}^{N} a_j g_j$ is still cts w/ compact support, and

$$\int |\phi - g|\,dm \le \sum_{j=1}^{N} |a_j| \int |\chi_{E_j} - g_j|\,dm$$
$$< \sum_{j=1}^{N} |a_j| \cdot \frac{\varepsilon}{2N|a_j|} = \frac{\varepsilon}{2}$$

Thus

$$\int |f-g|\,dm \le \int |f-\phi|\,dm + \int |\phi - g|\,dm < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \square$$

Rem: This theorem says cts, compactly supported functions are dense in the metric space $(L^1(m), \int |\cdot - \cdot|\,dm)$.

Def: For $f \in L^1(m)$ define its L^1-norm by

$$\|f\|_1 := \int |f|\,dm.$$

Note that $\|\cdot\|_1$ is indeed a norm:
~~That states that is~~

(i) $\|cf\|_1 = \int |cf|\,dm = |c| \int |f|\,dm = |c|\,\|f\|_1$, $c \in \mathbb{R}$

(ii) $\|f\|_1 = 0 \iff \int |f|\,dm = 0 \iff f = 0$ a.e. $\iff f = 0$ in $L^1(m)$

(iii) $\|f+g\|_1 = \int |f+g|\,dm \le \int |f| + |g|\,dm = \|f\|_1 + \|g\|_1$

Lemma: Let $(V, \|\cdot\|)$ be a normed vector space. Then it is complete iff every absolutely convergent series converges.

Pf: ($\Rightarrow$) Suppose $(V, \|\cdot\|)$ is complete. Let $(x_n)_{n\in\mathbb{N}} \subseteq V$ be s.t. the series

$$\sum_{n=1}^{\infty} \|x_n\|$$

converges. Then for $\varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t.

$$\sum_{n=N_0}^{\infty} \|x_n\| < \varepsilon.$$

Consequently, $\forall N, M \geq N_0$ (say $N \leq M$) we have:

$$\left\| \sum_{n=1}^{N} x_n - \sum_{n=1}^{M} x_n \right\| = \left\| \sum_{n=N+1}^{M} x_n \right\| \leq \sum_{n=N+1}^{M} \|x_n\| \leq \sum_{n=N_0}^{\infty} \|x_n\| < \varepsilon.$$

That is $\left(\sum_{n=1}^{N} x_n\right)_{N\in\mathbb{N}}$ is a Cauchy sequence, hence it converges to

$$\lim_{N\to\infty} \sum_{n=1}^{N} x_n = \sum_{n=1}^{\infty} x_n.$$

($\Leftarrow$) ~~Suppose~~ Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. It suffices to show $(x_n)_{n\in\mathbb{N}}$ has a conv. subsequence. ~~Indeed~~ We can find increasing integers

$$n_1 < n_2 < \cdots$$

s.t. $\|x_n - x_m\| < 2^{-k}$ $\forall n, m \geq n_k$. Define

$$y_1 = x_{n_1}$$
$$y_k = x_{n_k} - x_{n_{k-1}} \qquad k \geq 2$$

Observe that

$$\sum_{k=1}^{K} y_k = x_{n_1} + \sum_{k=2}^{K} (x_{n_k} - x_{n_{k-1}}) = x_{n_K}$$

Now,

$$\sum_{k=1}^{\infty} \|y_k\| \leq \|y_1\| + \sum_{k=2}^{\infty} 2^{-(k-1)} = \|y_1\| + 1 < \infty$$

Thus, by our hypothesis $\sum_{k=1}^{\infty} y_k$ converges. This means

$$\left(\sum_{k=1}^{K} y_k\right)_{K\in\mathbb{N}} = (x_{n_K})_{K\in\mathbb{N}}$$

converges. Thus $(x_n)_{n\in\mathbb{N}}$ converges and so $(V, \|\cdot\|)$ is complete $\square$

<u>Thm</u> $L^1(m)$ equipped with the L^1-norm is a Banach space: a complete, normed vector space

<u>Pf</u>: It was an exercise to show $L^1(m)$ is a vector space over $\mathbb{R}$, so it remains to show it is complete. By the Lemma, it suffices to consider abs. conv series. Let $(f_n)_{n\in\mathbb{N}} \subseteq L^1(m)$ be s.t.

$$B := \sum_{n=1}^{\infty} \|f_n\|_1 < \infty.$$

Define

$$G = \sum_{n=1}^{\infty} |f_n| \qquad G_N = \sum_{n=1}^{N} |f_n|.$$

Then by the MCT

$$\int G \, dm = \lim_{N\to\infty} \int G_N \leq B < \infty.$$

So $G \in L^1(m)$, and in particular $G < \infty$ a.e. Consequently $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. Set

$$F = \sum_{n=1}^{\infty} f_n$$

so that $|F| \leq G$, and hence $F \in L^1(m)$. Moreover for each $N \in \mathbb{N}$

$$\left| F - \sum_{n=1}^{N} f_n \right| \leq |F| + \sum_{n=1}^{N} |f_n| \leq 2|G| \in L^1(m)$$

So by the DCT

$$\lim_{N\to\infty} \left\| F - \sum_{n=1}^{N} f_n \right\|_1 = \lim_{N\to\infty} \int \left| F - \sum_{n=1}^{N} f_n \right| dm$$

$$= \int \lim_{N\to\infty} \left| F - \sum_{n=1}^{N} f_n \right| dm$$

$$= \int 0 \, dm = 0.$$

Thus $\left(\sum_{n=1}^{N} f_n\right)_{N\in\mathbb{N}}$ converges to F w.r.t. $\|\cdot\|_1$. $\square$