Modes of Convergence

Def: for $f, (f_n)_{n\in\mathbb{N}} \subseteq L^1(m)$, we say $f_n \to f$ in L^1 if
$$\|f_n - f\|_1 \to 0$$

we have seen that $f_n \to f$ a.e. does not imply $f_n \to f$ in L^1:

- $f_n = \chi_{(n,n+1)}$
- $f_n = n\cdot\chi_{(0,1/n)}$
- $f_n = \frac{1}{n}\cdot\chi_{(0,n)}$

It is also true that $f_n \to f$ in L^1 does not imply $f_n \to f$ a.e.

~~Ex/ Let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\cap[0,1]$ ordered so that $q_1 = \frac{0}{1}$~~

Type-writer Sequence

Ex: Define a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}\cap[0,1]$
$$x_0 = 0,\ x_1 = \tfrac{1}{1},\ x_2 = \tfrac{0}{2},\ x_3 = \tfrac{1}{2},\ x_4 = \tfrac{2}{2},\ x_5 = \tfrac{0}{3},\ x_6 = \tfrac{1}{3}, \ldots$$
Let
$$f_n := \begin{cases} \chi_{[x_{n-1},x_n]} & \text{if } x_{n-1} < x_n \\ \chi_{[x_n, x_{n+1}]} & \text{otherwise} \end{cases}$$

Then $\|f_n\|_1 = \int |f_n|\,dm = \frac{1}{m} \in \mathbb{Q}$ for some $m\in\mathbb{N}$, where $m\to\infty$ as $n\to\infty$. Thus $f_n \to 0$ in L^1. However, $\forall x\in[0,1]$ $\exists$ infinitely many $n\in\mathbb{N}$ s.t. $f_n(x) = 1$. So $f_n \not\to 0$ a.e.

Def: For $f, (f_n)_{n\in\mathbb{N}} \subseteq L^1(m)$, we say f_n converges to f in measure if $\forall \varepsilon > 0$
$$\lim_{n\to\infty} m(\{x\in\mathbb{R}^d : |f_n(x) - f(x)| > \varepsilon\}) = 0$$

<u>Prop</u>: If $f_n \to f$ in L^1, then $f_n \to f$ in measure.

<u>Pf</u>: Define
$$E_{n,\varepsilon} := \{x \in \mathbb{R}^d : |f_n(x) - f(x)| \geq \varepsilon\}.$$
Then
$$\int |f_n - f|\, dm \geq \int_{E_{n,\varepsilon}} |f_n - f|\, dm \geq \int_{E_{n,\varepsilon}} \varepsilon \cdot dm$$
$$= \varepsilon \cdot m(E_{n,\varepsilon})$$

Thus $m(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon} \int |f_n - f|\, dm$. Thus
$$\lim_{n\to\infty} m(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon} \lim_{n\to\infty} \|f_n - f\|_1 = 0. \quad \square$$

<u>Thm</u>: Suppose $f_n \to f$ in measure. Then $\exists$ a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ s.t. $f_{n_k} \to f$ a.e.

<u>Pf</u>: For each $k \in \mathbb{N}$, choose n_k s.t. if
$$E_k := \{x : |f_{n_k} - f| \geq 2^{-k}\}$$
Then $m(E_k) \leq 2^{-k}$. Since $\sum m(E_k) \leq \sum 2^{-k} < \infty$, the Borel-Cantelli lemma (from HW) implies
$$m\Big(\bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k\Big) = 0$$

Set $S :=$ ~~Then for $x \in S^c$, we have~~

$$S^c = \bigcup_{N\in\mathbb{N}} \bigcap_{k\geq N} \{x : |f_{n_k} - f| < 2^{-k}\}$$

So for $x \in S^c$, $\exists N \in \mathbb{N}$ st $\forall k \geq N$ $|f_{n_k}(x) - f(x)| < 2^{-k}$

Thus
$$\lim_{k\to\infty} |f_{n_k}(x) - f(x)| \leq \lim_{k\to\infty} 2^{-k} = 0.$$

That is $f_{n_k} \to f$ on $S^c \Rightarrow$ a.e. $\square$

<u>Cor</u>: If $f_n \to f$ in L^1, then $\exists (f_{n_k})$ subseq. s.t. $f_{n_k} \to f$ a.e.

<u>Exercise</u>: Find such a subsequence for the type-writer sequence.

Thm (Egorov's Thm)

Let $E \in \mathcal{M}$ be s.t. $m(E) < \infty$. Suppose $f_n \to f$ a.e. on E. Then $\forall \varepsilon > 0$, $\exists\, F \subseteq E$ s.t. $m(F) < \varepsilon$ and $f_n \to f$ uniformly on $E \setminus F$.

Pf: By modifying f_n, f on a ctble union of zero sets (i.e. a zero set) we may assume $f_n \to f$ everywhere on E.

For each $n, k \in \mathbb{N}$, define

$$E_n(k) := \bigcup_{j=n}^{\infty} \{x : |f_j(x) - f(x)| \geq \tfrac{1}{k}\}.$$

Then for fixed k,

$$E_1(k) \supseteq E_2(k) \supseteq E_3(k) \supseteq \cdots$$

and

$$\bigcap_{n=1}^{\infty} E_n(k) = \emptyset$$

since $x \in \bigcap$ iff $\forall n \in \mathbb{N}\ \exists\, j \geq n$ s.t. $|f_j(x) - f(x)| \geq \frac{1}{k}$, that is $f_n(x) \not\to f(x)$. Since $m(E_1(k)) \leq m(E) < \infty$, cty from above implies

$$\lim_{n \to \infty} m(E_n(k)) = 0.$$

Let $\varepsilon > 0$ and let $k \in \mathbb{N}$ be arbitrary. Choose $n_k \in \mathbb{N}$ large enough s.t. $m(E_{n_k}(k)) < \varepsilon \cdot 2^{-k}$.

Set

$$E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$$

So that

$$m(E) \leq \sum_{k=1}^{\infty} m(E_{n_k}(k)) \leq \sum_{k=1}^{\infty} \varepsilon 2^{-k} = \varepsilon.$$

Also

$$E^c = \bigcap_{k=1}^{\infty} E_{n_k}(k)^c = \bigcap_{k=1}^{\infty} \bigcap_{j=n_k}^{\infty} \{x : |f_j(x) - f(x)| < \tfrac{1}{k}\}$$

So for $x \in E^c$, $\forall k \in \mathbb{N}\ \forall j \geq n_k$ $|f_j(x) - f(x)| < \frac{1}{k}$.

Thus $\forall \delta > 0$, letting $\frac{1}{k} < \delta$, $\forall j \geq n_k$ we have

$$|f_j(x) - f(x)| < \tfrac{1}{k} < \delta.$$

That is, $f_n \to f$ unif. on E^c. $\square$