Outer Measure of a Closed Box

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Proposition. *Let*

$$A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n.$$

Then

$$m^*(A) = \prod_{i=1}^{n} (b_i - a_i).$$

Proof. Let $\epsilon > 0$, and observe that if

$$B := (a_1 - \epsilon, b_1 + \epsilon) \times \cdots \times (a_n - \epsilon, b_n + \epsilon),$$

then $\{B\}$ is a countable covering of A by open boxes. Thus

$$m^*(A) \leq |B| = \prod_{i=1}^{n} (b_i - a_i + 2\epsilon).$$

Letting $\epsilon \to 0$ yields $m^*(A) = \prod (b_i - a_i)$.

Now, let $\{B_k\}_{k \in \mathbb{N}}$ be a countable covering of A by open boxes. Since A is a closed and bounded subset of $\mathbb{R}^n$, it is compact by the Heine-Borel theorem. Hence the open covering $\{B_k\}_{k \in \mathbb{N}}$ can be reduced to a finite subcover $\{B_{k_1}, \ldots, B_{k_d}\}$. Observe that

$$\chi_A \leq \chi_{B_{k_1}} + \cdots + \chi_{B_{k_d}}.$$

Thus we have

$$\begin{aligned}
\prod_{i=1}^{n} (b_i - a_i) &= \int \chi_A && \text{(Fubini's theorem)} \\
&\leq \sum_{j=1}^{d} \int \chi_{B_{k_j}} && \text{(monotonicity of the Riemann integral)} \\
&= \sum_{j=1}^{d} |B_{k_j}| && \text{(Fubini's theorem again)} \\
&\leq \sum_{k=1}^{\infty} |B_k|.
\end{aligned}$$

Since $\{B_k\}_{k \in \mathbb{N}}$ was an arbitrary countable covering of A by open boxes, we have $\prod (b_i - a_i) \leq m^*(A)$ and the desired equality follows. □