A European application of random matrix theory

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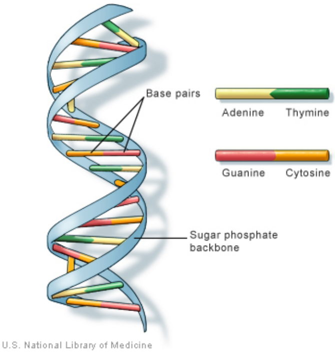
Michigan State University

Math 317H
December 5th, 2019

Genetics

[Novembre et al., Nature 2008]: Researchers analyzed genetic data from people with European ancestry: 1,387 people at 197,146 genetic loci.



$$\begin{array}{ccc} A & \mapsto & 1 \\ C & \mapsto & 2 \\ G & \mapsto & 3 \\ T & \mapsto & 4 \end{array}$$

They recorded these numbers as a 1387×197146 matrix X.

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They then analyzed this data by examining the singular values of X....

Probability Theory

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Unfair Coins:

https://izbicki.me/blog/how-to-create-an-unfair-coin-and-prove-it-with-math.html

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Probability theory: Provides the tools needed to compute the expected outcome.

Random Matrix Theory

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Flip a coin, define $x = 1$ if the coin comes up heads and $x = -1$ if the coin comes up tails. Roll a 6-sided die and let y be the result. Then

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \qquad B = \begin{pmatrix} 1 & y \\ y & 1 \end{pmatrix}$$

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Since the matrix is (at least partially) random, the data associated to the matrix is potentially random as well: the entries of X, $\det(X)$, $\mathrm{Tr}(X)$, eigenvalues of X, eigenvectors of X, etc.

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Least Squares? (continued)

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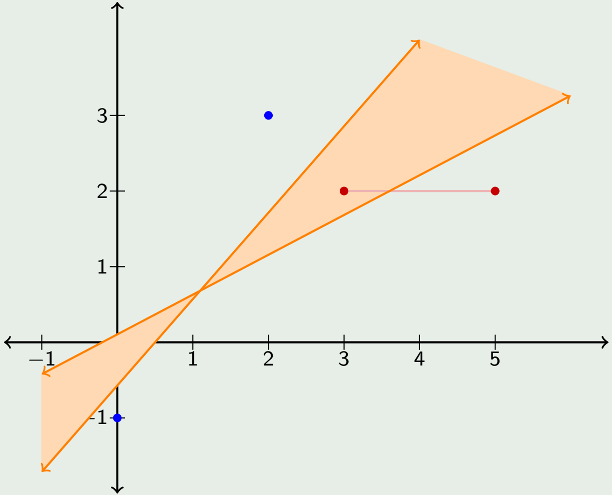

Not McConnell

McConnell

Eugene Wigner

Sad McConnell

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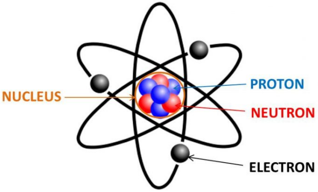

http://www.whoinventedfirst.com/who-discovered-the-atom/

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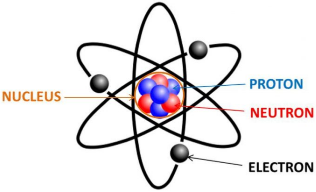

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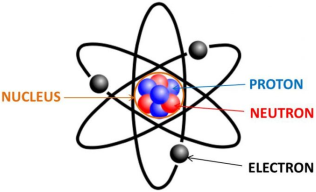

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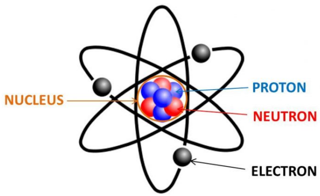

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- But for atoms with "heavy nuclei" (e.g uranium-238: 92 protons, 92 electrons, 146 neutrons), too complicated to solve explicitly
- **Wigner's idea:** treat the Hamiltonian as a random matrix

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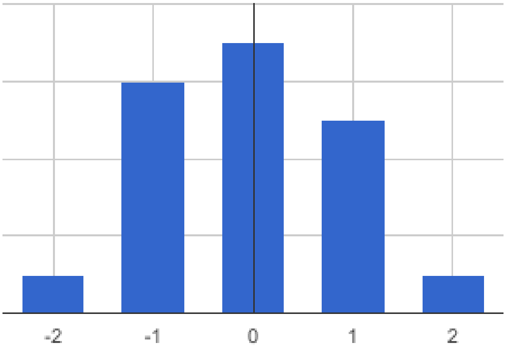

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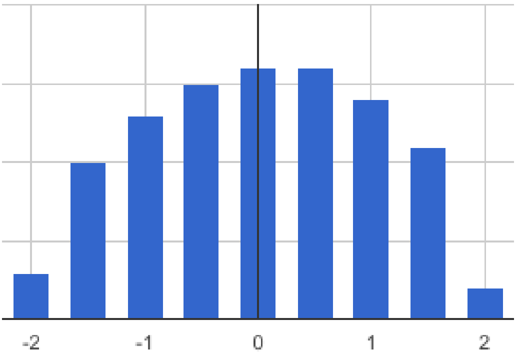

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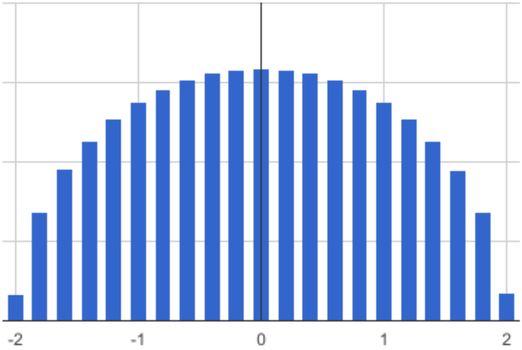
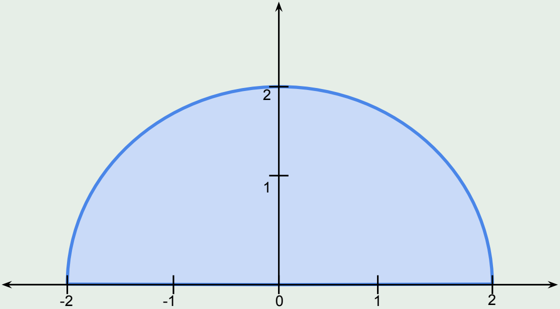


$n = 10,000$

Semicircle Distribution

The histograms get closer and closer to the **semicircle distribution**:

$$s(t) = \begin{cases} \frac{1}{2\pi}\sqrt{4-t^2} & \text{if } -2 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$



Theorem (Wigner's Semicircle Law)

Let X_n, $n \in \mathbb{N}$, be the sequence of symmetric random matrices as above. For any interval $[a, b] \subset \mathbb{R}$,

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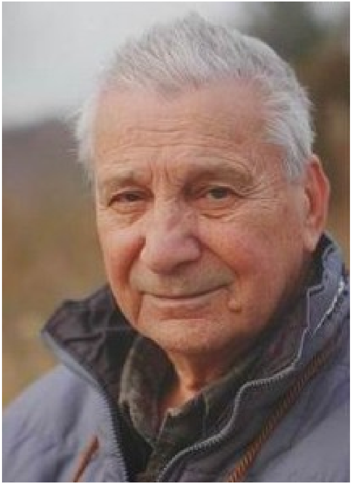
That is, there exists an $N \in \mathbb{N}$ so that for any $n \geq N$

$$\frac{\#\{\text{eigenvalues of } X_n \text{ in the interval } [a,b]\}}{n} \approx \frac{1}{2\pi} \int_a^b \sqrt{4 - t^2}\ dt.$$

The Marčenko–Pastur Law



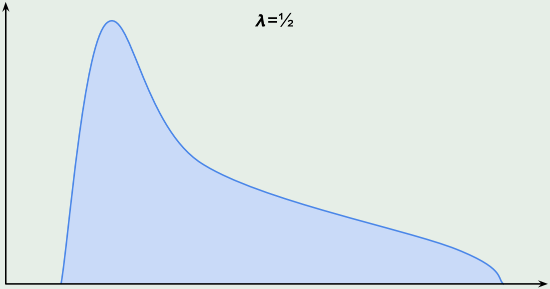
Vladimir Marčenko

Leonid Pastur

Marčenko–Pastur distribution

Fix $\lambda \in (0,1]$ and define $\lambda_\pm = (1 \pm \sqrt{\lambda})^2$. The **Marčenko–Pastur distribution** is

$$\nu(t) := \begin{cases} \frac{1}{2\pi\lambda} \frac{\sqrt{(t-\lambda_-)(\lambda_+-t)}}{t} & \text{if } \lambda_- \leq t \leq \lambda_+ \\ 0 & \text{otherwise} \end{cases}.$$



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Theorem (The Marčenko–Pastur Law)

With Y_n, $n \in \mathbb{N}$, as above, for any interval $[a,b] \subset \mathbb{R}$,

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Back to Genetics

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Set

$$
\begin{aligned} n &:= 197,146 \\ p(n) &:= 1,387 \end{aligned} \qquad \lambda := \frac{p(n)}{n} = \frac{1,387}{197,146} \approx 0.007035
$$

They computed eigenvalues of $Y := \frac{1}{n}XX^T$, which is a $p(n) \times p(n)$ symmetric matrix.

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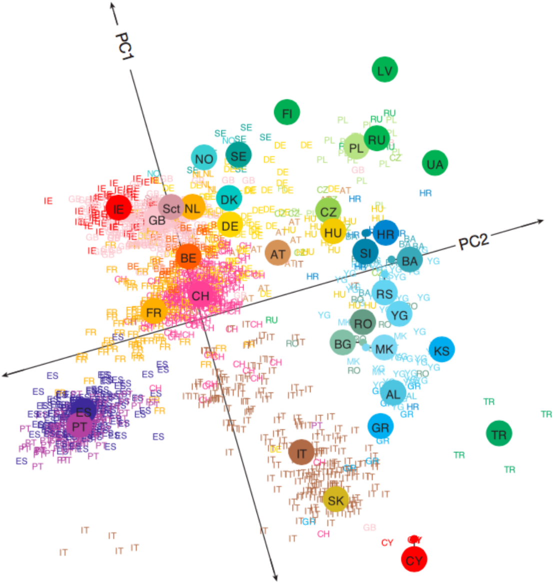
- Something pretty incredible happens when you plot these coordinate pairs...


PC1
PC2
LV
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UA
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Sct
NL
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DE
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HU
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SI
BA
BE
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CH
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FR
YG
RO
BG
MK
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AL
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TR
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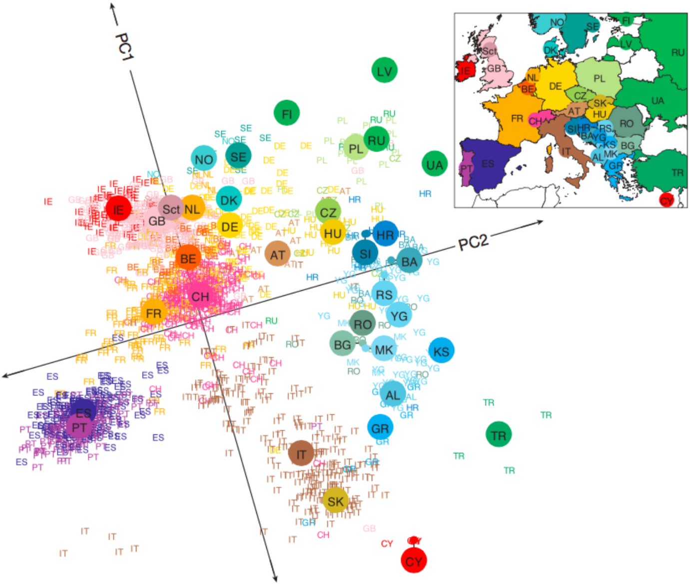
SK
CY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