9/2

1.0 General Notation

- $\mathbb{R}$ - real numbers
- $\mathbb{C}$ - complex numbers
- $\mathbb{N}$ - natural numbers $(1, 2, 3, \ldots)$
- $\mathbb{Z}$ - integers $(\ldots, -2, 1, 0, 1, 2, \ldots)$
- $\mathbb{Q}$ - rational numbers $(\frac{3}{4}, -\frac{2}{7}, \text{etc.})$
- $x \in \mathbb{R}$ "x is an element of $\mathbb{R}$" or "x is in the set $\mathbb{R}$"
- Set Notation:
  - $\{\underbrace{x \in \mathbb{Z}}_{\text{type of elements}} \mid \underbrace{x \geq 1}_{\text{conditions}}\} = \mathbb{N}$

    "such that"
  - $\{x \in \mathbb{R} \mid x = \frac{n}{m} \text{ for some } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}\} = \mathbb{Q}$
- $A \subset B$ "the set A is contained in the set B" or "the set A is a subset of the set B"
  - $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
  - $\mathbb{R} \subset \mathbb{R}$, $\mathbb{Q} \subsetneq \mathbb{R}$ "$\mathbb{Q}$ is contained in but not equal to $\mathbb{R}$" (strict subset).

1.1 Vector Spaces

Def A vector space V is a collection of objects (called vectors) equipped with operations of addition and scalar multiplication such that the following vector space axioms hold:

Addition

1. Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for all $\vec{v}, \vec{w} \in V$
2. Associativity: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$;
3. Zero Vector: there exists a special vector, denoted by $\vec{0}$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$. It is called the zero vector;
4. Additive Inverse: for every vector $\vec{v} \in V$ there exists a vector $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$. This vector is called the additive inverse of $\vec{v}$ and is usually denoted $-\vec{v}$;

Scalar multiplication

5. Multiplicative Identity: $1\vec{v} = \vec{v}$ for all $\vec{v} \in V$;
6. Multiplicative associativity: $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$ for all $\vec{v} \in V$ and all scalars α, β;

distributive laws

7. $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ for all $\vec{u}, \vec{v} \in V$ and scalars α;
8. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in V$ and all scalars α, β.

- Vectors can potentially be very abstract objects (we'll see some examples soon), whereas "scalars" is just a fancy term for numbers. Sometimes we will mean the real numbers $\mathbb{R}$ and sometimes we'll mean the complex numbers $\mathbb{C}$. We call V a real vector space in the former case, and a complex vector space in the latter case. If we do not specify $\mathbb{R}$ or $\mathbb{C}$ or we write $\mathbb{F}$ then the statement holds for both. (It may even hold for any "field" $\mathbb{F}$, but we'll focus on $\mathbb{R}$ and $\mathbb{C}$ in this course).

- It is important to distinguish between vectors and scalars. So vectors will always be decorated with an arrow: $\vec{v}$ (or will be bold when typed). Scalars will usually be Greek letters $(\alpha, \beta, \gamma \ldots)$ while vectors will be roman letters $(u, v, w, \ldots)$.

$\boxed{\text{Remark}}$ The above axioms should be (at least vaguely) familiar from algebra/arithmetic, where they apply to just numbers rather than vectors and scalars. Consequently you should not need memorize the axioms (and in particular I will not ask you to do so), but you do need to remember what operations apply to what objects. For example, you can add vectors, and multiply a vector by a scalar, but you cannot multiply two vectors. $\cancel{\vec{u}\,\vec{v}}$

$\boxed{\text{Examples}}$

0. $V = \mathbb{R}$ is a real vector space where the vectors are just real numbers and so are the scalars, so all the axioms trivially hold. Similarly $V = \mathbb{C}$ is a complex vector space. We can also make it a real vector space, since a real number times a complex number is still a complex number.

1. For $n \in \mathbb{N}$, $n \geq 2$ let $\mathbb{R}^n$ denote the columns with n entries from $\mathbb{R}$:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Then $\mathbb{R}^n$ is a vector space entry-wise operations:

$$\underbrace{\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix} \in \mathbb{R}^n}_{\text{addition}} \qquad \underbrace{\alpha \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix} \in \mathbb{R}^n}_{\text{scalar multiplication}}$$

Let's check a few axioms:

Commutativity:
$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix} \underset{\text{since addition in } \mathbb{R} \text{ is commutative}}{=} \begin{pmatrix} w_1+v_1 \\ w_2+v_2 \\ \vdots \\ w_n+v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Zero vector: we claim $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Indeed

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v_1+0 \\ v_2+0 \\ \vdots \\ v_n+0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Multiplicative Associativity:

$$(\alpha\beta)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\alpha\beta)v_1 \\ (\alpha\beta)v_2 \\ \vdots \\ (\alpha\beta)v_n \end{pmatrix} = \begin{pmatrix} \alpha(\beta v_1) \\ \alpha(\beta v_2) \\ \vdots \\ \alpha(\beta v_n) \end{pmatrix} = \alpha\begin{pmatrix} \beta v_1 \\ \beta v_2 \\ \vdots \\ \beta v_n \end{pmatrix} = \alpha\left(\beta\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\right).$$

$\mathbb{C}^n$ is defined similarly but with complex entries, and it is a complex vector space.

2) For $n \in \mathbb{N}$, let $\mathbb{P}_n$ denote the collection of polynomials of degree at most n:

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{P}_n$$

Note that the coefficients a_i are allowed to be zero.
Addition and scalar multiplication are given by:

$$(a_n t^n + \cdots + a_1 t + a_0) + (b_n t^n + \cdots + b_1 t + b_0) = (a_n + b_n)t^n + \cdots + (a_1+b_1)t + (a_0+b_0)$$

$$\alpha(a_n t^n + \cdots + a_1 t + a_0) = (\alpha a_n)t^n + \cdots + (\alpha a_1)t + (\alpha a_0)$$

If we consider only real coefficients, then $\mathbb{P}_n$ is a real vector space. If we allow complex coefficients, then it is a complex vector space.

What is $\vec{0}$ here? $p(t) = 0t^n + \cdots + 0t + 0 = 0.$

What is the additive inverse of $p(t) = 3t^3 - t^2 + 4it + 1.2$? $-p(t) = -3t^3 + t^2 - 4it - 1.2$?

3) (Netflix)

List genres of movies: $g_1, g_2, \ldots, g_N$.
Consider the vector space

$$V = \mathbb{R}^N$$

with the usual operations of scalar mult. and addition.
For a netflix user, define $x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in V$ for each $i = 1, \ldots, N$ by:

x_i = (# of movies of genre g_i they have finished watching) − (# of movies in genre g_i they started but didn't finish)

Heuristically: x_i is positive if they like movies in genre g_i.
Netflix will use this data to recommend movies to you.

Addition $x+y$ corresponds to user x and user y sharing an account.

$\square$

Rem It is implicit in the definition of a vector space V that it is "closed" under the addition and scalar multiplication operations. That is:

1. For any $\vec{v}, \vec{w} \in V$ we must have that $\vec{v}+\vec{w}$ is <u>also</u> in V.
2. For any $\vec{v} \in V$ and any scalar α, we must have that $\alpha\vec{v}$ is <u>also</u> in V.

In particular, this means that if a set V fails to be closed under either of these operations, then it is <u>not</u> a vector space.

EX Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2+y^2=z^2 \right\}$$

with addition and scalar multiplication defined as for $\mathbb{R}^3$. Then V is <u>not</u> a vector space because it is not closed under addition. Indeed,

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are both in V, but $\vec{v}+\vec{w} = (1,1,2)^T$ is <u>not</u> since $1^2+1^2 \neq 2^2$. $\square$

Theorem 1.1 For every vector space V, the zero vector $\vec{0}$ is unique.

Proof Suppose $\vec{0}$ and $\vec{0}'$ both satisfy the zero vector axiom:

$$\vec{v}+\vec{0}=\vec{v} \quad \text{and} \quad \vec{v}+\vec{0}'=\vec{v} \quad \text{for all } \vec{v}\in V.$$

Then we have

$$\begin{aligned} \vec{0}' &= \vec{0}'+\vec{0} && \text{(first equation)} \\ &= \vec{0}+\vec{0}' && \text{(commutativity)} \\ &= \vec{0} && \text{(second equation)} \end{aligned}$$

$\square$

Thm 1.2 Let V be a vector space. For any $\vec{v}\in V$, $0\vec{v}=\vec{0}$.

Proof Fix $\vec{v}\in V$. First observe

$$\vec{v} = 1\vec{v} = (1+0)\vec{v} = 1\vec{v}+0\vec{v} = \vec{v}+0\vec{v}$$

So $\vec{v}=\vec{v}+0\vec{v}$. But then adding $-\vec{v}$ to each side we get:

$$\begin{aligned} \vec{v}+-\vec{v} &= \vec{v}+0\vec{v}+-\vec{v} \\ \vec{0} &= \vec{v}+-\vec{v}+0\vec{v} \\ \vec{0} &= \vec{0}+0\vec{v} \\ \vec{0} &= 0\cdot\vec{v} \end{aligned}$$

as claimed. $\square$

9/7

Matrix Notation

An $m\times n$ <u>matrix</u> is a rectangular array with m rows and n columns. Elements in the array are called entries and can be <u>real</u> or <u>complex</u> numbers.

$\boxed{\text{Ex}}$

$\begin{pmatrix} 2 & -1 \\ 0 & 4.3 \\ 16 & -0.\bar{3} \end{pmatrix}$ is 3×2 $\quad\begin{pmatrix} 2+i & e^{\pi i} & 1 \end{pmatrix}$ is 1×3 $\square$

- $M_{m\times n}(\mathbb{R})$: $m\times n$ matrices with real entries

$M_{m\times n}(\mathbb{C})$: $m\times n$ matrices with complex entries

$M_{m\times n}$ refers to both. Using entrywise addition and scalar mult. (like $\mathbb{R}^n$ and $\mathbb{C}^n$) $M_{m\times n}(\mathbb{R})$ is a real vector space and $M_{m\times n}(\mathbb{C})$ is a complex vector space.

- For $A\in M_{m\times n}$, we write $(A)_{jk}$ for the entry of A in row j and column k, with $j=1,\dots,m$ and $k=1,\dots,n$.

$$A=\begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 5 & -2 \end{pmatrix} \qquad (A)_{21}=2 \qquad (A)_{32}=-2 \qquad (A)_{23}=\text{DNE}$$

We may also use lower-case letters to denote entries: $a_{jk}=(A)_{jk}$. In this case we could define A by writing

$$A=(a_{jk})_{j=1,k=1}^{m\quad n}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$\boxed{\text{Def}}$ For a matrix $A\in M_{m\times n}$, its transpose, denoted A^T, is the $n\times m$ matrix satisfying $(A^T)_{jk}=(A)_{kj}$ for $j=1,\dots,n$ and $k=1,\dots,m$.

Taking the transpose of a matrix turns its rows into columns (and vice-versa)

$$A=\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \qquad A^T=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$B=\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \qquad B^T=\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

- Note that $\mathbb{F}^n$ is the same as $M_{n\times 1}(\mathbb{F})$. So we can take the transpose of a vector:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}^T=(v_1, v_2, \dots, v_n) \leftarrow \text{row vector}$$

$\leftarrow$ column vector

Also $(x_1, x_2, \dots, x_n)^T=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\in\mathbb{F}^n$. Since this saves space, we will frequently write elements of $\mathbb{F}^n$ as the transpose of a row vector.

$\boxed{1.2}$ Linear combinations, bases

Let's make use of our two operations: addition and scalar multiplication

$\boxed{\text{Def}}$ Let $\vec{v}_1, \ldots, \vec{v}_p \in V$ be a collection of vectors. A linear combination of $\vec{v}_1, \ldots, \vec{v}_p$ is a sum of the form

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_p \vec{v}_p \quad \left(= \sum_{k=1}^{p} \alpha_k \vec{v}_k \right)$$

where $\alpha_1, \ldots, \alpha_p$ are scalars. We call the linear combination trivial if $\alpha_1 = \cdots = \alpha_p = 0$ (in which case the above is $\vec{0}$), and otherwise it is non-trivial. The set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_p$

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_p\} := \left\{ \sum_{k=1}^{p} \alpha_k \vec{v}_k : \alpha_1, \ldots, \alpha_k \text{ scalars} \right\}$$

is called the span of $\vec{v}_1, \ldots, \vec{v}_p$. $\square$

Note that since a vector space V is closed under addition and scalar multiplication, any linear combination of $\vec{v}_1, \ldots, \vec{v}_p$ will be some vector in V. Consequently $\text{span}\{\vec{v}_1, \ldots, \vec{v}_p\} \subset V$.

$\boxed{\text{Ex}}$ For $V = \mathbb{F}^n$ (recall $\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{F}$), consider the vectors:

$$\vec{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \vec{e}_n := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Then $\text{span}\{\vec{e}_1, \ldots, \vec{e}_n\} = \mathbb{F}^n$. Indeed, for any $\vec{v} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{F}^n$ we have

$$\vec{v} = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n \in \text{span}\{\vec{e}_1, \ldots, \vec{e}_n\}.$$

So $\mathbb{F}^n \subset \text{span}\{\vec{e}_1, \ldots, \vec{e}_n\}$ and the reverse inclusion is immediate. Also note that the above linear combination is the only way to obtain $\vec{v}$: if $\vec{v} = y_1 \vec{e}_1 + \cdots + y_n \vec{e}_n$ for some scalars $y_1, \ldots, y_n$ then

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{v} = y_1 \vec{e}_1 + \cdots + y_n \vec{e}_n = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

and therefore $x_i = y_i$ for $i = 1, \ldots, n$. So $x_1, \ldots, x_n$ is the unique choice of scalars that gives $\vec{v}$.

We also claim

$$\text{span}\{\vec{e}_2, \ldots, \vec{e}_n\} = \left\{ \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_2, \ldots, x_n \in \mathbb{F} \right\}$$

Indeed, for scalars $\alpha_1, \ldots, \alpha_{n-1}$ we have

$$\alpha_1 \vec{e}_2 + \cdots + \alpha_{n-1} \vec{e}_n = \begin{pmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

and for $\vec{v} = (0, x_2, \ldots, x_n)^T$ we have $\vec{v} = x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n \in \text{span}\{\vec{e}_2, \ldots, \vec{e}_n\}$. $\square$

In this section, we are interested in answering the following questions for a fixed collection of vectors $\vec{v}_1, \dots, \vec{v}_p \in V$:

1. Can every vector in V be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$?
2. For a vector $\vec{v} \in V$ that is a linear combination of $\vec{v}_1, \dots, \vec{v}_p$, is there a unique linear combination that gives $\vec{v}$? (i.e. is there only one choice of scalars $\alpha_1, \dots, \alpha_p$ so that $\sum \alpha_k \vec{v}_k = \vec{v}$?)

When the answer to both of these questions is "Yes" — such as in the first part of the previous example — we have a "basis". However, it can be that only one or even neither of these questions has an affirmative answer: for $\vec{e}_2, \dots, \vec{e}_n$ as in the previous example the answer to 1 is "No" while the answer to 2 is "Yes" (Exercise: check this). Thus we must investigate these questions independently, and we will start with 1 where we note an answer of "Yes" is equivalent to $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_p\} = V$.

$\boxed{\text{Def}}$ A system of vectors $\vec{v}_1, \dots, \vec{v}_p \in V$ is called a spanning (or generating) system if $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_p\} = V$. That is, every vector of V can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$. In this case we also say the system spans V. $\square$

$\boxed{\text{EX}}$ 1. $\vec{e}_1, \dots, \vec{e}_n \in \mathbb{F}^n$ as in the example above span $\mathbb{F}^n$, while $\vec{e}_2, \dots, \vec{e}_n$ do not.

2. Let $\mathbb{P}_2$ be the space of polynomials of degree at most 2, and define
$$\vec{v}_1 := t^2 - t, \qquad \vec{v}_2 := t - 1, \qquad \vec{v}_3 := 1$$
Then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a spanning system: for $p(t) = a_2 t^2 + a_1 t + a_0$ we have
$$\begin{aligned} p(t) &= a_2(t^2 - t) + (a_1 + a_2)t + a_0 \\ &= a_2 \vec{v}_1 + (a_1 + a_2)(t-1) + (a_0 + a_1 + a_2) 1 \\ &= a_2 \vec{v}_1 + (a_1 + a_2)\vec{v}_2 + (a_0 + a_1 + a_2)\vec{v}_3 \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}. \end{aligned}$$

3. If $\vec{v}_1, \dots, \vec{v}_p \in V$ span V, then for any additional vectors $\vec{v}_{p+1}, \dots, \vec{v}_r \in V$, the system $\vec{v}_1, \dots, \vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_r$ spans V. Indeed, for scalars $\alpha_1, \dots, \alpha_p$ observe that
$$\alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p + 0 \cdot \vec{v}_{p+1} + \dots + 0 \cdot \vec{v}_r \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_r\}.$$
Hence
$$V = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_p\} \subset \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_r\} \subset V,$$
which implies $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_r\} = V$. $\square$

In Chapter 2 we will develop methods for determining if a system of vectors in $\mathbb{F}^n$ is a spanning system. For now, we turn towards 2 above.

Theorem 1.3 Let $\vec{v}_1, \dots, \vec{v}_p \in V$ be a system of vectors in a vector space V. Then every $\vec{v} \in \text{span}\{\vec{v}_1, \dots \vec{v}_p\}$ can be written as a unique linear combination of $\vec{v}_1, \dots, \vec{v}_p$ if and only if whenever

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p = \vec{0}$$

for some scalars $\alpha_1, \dots, \alpha_p$ one has $\alpha_1 = \cdots = \alpha_p = 0$.

Proof ($\Longrightarrow$): Observe that

$$\vec{0} = 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_p \in \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}.$$

By assumption this is the unique linear combination that gives $\vec{0}$, and so whenever

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p = \vec{0}$$

we must have $\alpha_1 = \cdots = \alpha_p$.

($\Longleftarrow$): We will proceed by contrapositive, so assume there exists $\vec{v} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$ that cannot be written uniquely as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$. That is, there exists scalars $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p$ such that $\alpha_j \neq \beta_j$ for at least one $1 \leq j \leq p$ and

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p = \vec{v} = \beta_1 \vec{v}_1 + \cdots + \beta_p \vec{v}_p.$$

Subtracting the right side from the left side gives

$$\begin{aligned} \vec{0} &= (\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p) - (\beta_1 \vec{v}_1 + \cdots + \beta_p \vec{v}_p) \\ &= (\alpha_1 - \beta_1) \vec{v}_1 + \cdots + (\alpha_p - \beta_p) \vec{v}_p, \end{aligned}$$

and the coefficients $\alpha_j - \beta_j$ above are not all zero since $\alpha_j \neq \beta_j$ for at least one $1 \leq j \leq p$. □

Remark Note that the previous theorem implies that as soon as $\vec{0}$ can be written uniquely as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$, then so can every $\vec{v}$ in their span. □

Thus we see that the answer to 2 above is "Yes" precisely when the only linear combination of $\vec{v}_1, \dots, \vec{v}_p$ that gives $\vec{0}$ is the trivial linear combination. In light of this we make the following definition.

Def A system of vectors $\vec{v}_1, \dots, \vec{v}_p \in V$ is called linearly independent if whenever

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p = \vec{0}$$

for some scalars $\alpha_1, \dots, \alpha_p$ then one has $\alpha_1 = \cdots = \alpha_p = 0$. We say the system is linearly dependent if it is not linearly independent; that is, if

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p = \vec{0}$$

for some scalars $\alpha_1, \dots, \alpha_p$ which are not all zero. □

Ex ① $\vec{e}_1, \dots, \vec{e}_n \in \mathbb{F}^n$ from the first example in this section are linearly independent since we showed any $\vec{v} \in \mathbb{F}^n = \text{span}\{\vec{e}_1, \dots, \vec{e}_n\}$ can be written uniquely as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

(2) Let V be the set of functions $f: \mathbb{R} \to \mathbb{R}$ with operations

$$(f+g)(x) = f(x) + g(x) \qquad (\alpha f)(x) = \alpha f(x)$$

Then V is a real vector space whose zero vector is given by the zero function: $f_0(x) = 0$ $\forall x \in \mathbb{R}$ (Exercise check the axioms).

We claim

$$f_1(x) := 2 \qquad f_2(x) := x^2(x-1) \qquad f_3(x) := 1 - e^x$$

are linearly independent in V. Indeed, suppose there are scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) = f_0(x)$$
$$\alpha_1 (2) + \alpha_2(x^2(x-1)) + \alpha_3(1-e^x) = 0$$

Then plugging in $x=0$ gives

$$\alpha_1(2) + \alpha_2(0) + \alpha_3(0) = 0 \implies \alpha_1 = 0.$$

Thus $\alpha_2(x^2(x-1)) + \alpha_3(1-e^x) = 0$ and plugging in $x=1$ gives

$$\alpha_2(0) + \alpha_3(1-e^1) = 0 \implies \alpha_3 = 0.$$

Thus $\alpha_3(x^2(x-1)) = 0$ and plugging in any $x \notin \{0,1\}$ gives $\alpha_2 = 0$. So $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and hence $f_1(x), f_2(x), f_3(x)$ are linearly independent.

(3) If $\vec{v}_1, \ldots, \vec{v}_p \in V$ are linearly independent, then any subsystem of vectors $\vec{v}_{j_1}, \ldots, \vec{v}_{j_q}$ with $1 \le j_1 < \cdots < j_q \le p$ is linearly independent. See Exercise 2 on Homework 2. □

Proposition 1.4 A system of vectors $\vec{v}_1, \ldots, \vec{v}_p \in V$ is linearly dependent if and only if one the vectors $\vec{v}_k$ can be written as a linear combination of the others:

$$\vec{v}_k = \beta_1 \vec{v}_1 + \cdots + \beta_{k-1}\vec{v}_{k-1} + \beta_{k+1}\vec{v}_{k+1} + \cdots + \beta_p \vec{v}_p$$

for some scalars β_j, $j = 1, \ldots, k-1, k+1, \ldots, p$.

Proof ($\Longrightarrow$) Assume $\vec{v}_1, \ldots, \vec{v}_p$ are linearly dependent so that

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p = 0$$

for scalars $\alpha_1, \ldots, \alpha_p$ which are not all zero. Let $1 \le k \le p$ be such that $\alpha_k \ne 0$. Then adding $-\alpha_k \vec{v}_k$ to both sides and dividing by $-\alpha_k$ ($\ne 0$) yields

$$\frac{-\alpha_1}{\alpha_k}\vec{v}_1 + \cdots + \frac{-\alpha_{k-1}}{\alpha_k}\vec{v}_{k-1} + \frac{-\alpha_{k+1}}{\alpha_k}\vec{v}_{k+1} + \cdots + \frac{-\alpha_p}{\alpha_k}\vec{v}_p = \vec{v}_k$$

So we take $\beta_j := \frac{-\alpha_j}{\alpha_k}$ for $j = 1, \ldots, k-1, k+1, \ldots, p$.

($\Longleftarrow$) Suppose

$$\vec{v}_k = \beta_1 \vec{v}_1 + \cdots + \beta_{k-1}\vec{v}_{k-1} + \beta_{k+1}\vec{v}_{k+1} + \cdots + \beta_p \vec{v}_p.$$

for some scalars β_j. The subtracting $\vec{v}_k$ from each side yields:

$$\vec{0} = \beta_1 \vec{v}_1 + \cdots + \beta_{k-1}\vec{v}_{k-1} + (-1)\vec{v}_k + \beta_{k+1}\vec{v}_{k+1} + \cdots + \beta_p \vec{v}_p.$$

Since the coefficients are not all zero ($-1 \ne 0$ for example), $\vec{v}_1, \ldots, \vec{v}_p$ are linearly dependent. □

Ex If $\vec{v}_1, \dots, \vec{v}_p \in V$ are linearly dependent, then for any additional vectors $\vec{v}_{p+1}, \dots, \vec{v}_r \in V$ the system $\vec{v}_1, \dots, \vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_r$ is linearly dependent. Indeed, the linear dependence of $\vec{v}_1, \dots, \vec{v}_p$ and the previous proposition imply

$$\vec{v}_k = \beta_1 \vec{v}_1 + \dots + \beta_{k-1} \vec{v}_{k-1} + \beta_{k+1} \vec{v}_{k+1} + \dots + \beta_p \vec{v}_p$$

for some $1 \le k \le p$ and some scalars. Hence

$$\vec{v}_k = \beta_1 \vec{v}_1 + \dots + \beta_{k-1} \vec{v}_{k-1} + \beta_{k+1} \vec{v}_{k+1} + \dots + \beta_p \vec{v}_p + 0\,\vec{v}_{p+1} + \dots + 0 \cdot \vec{v}_r$$

and so $\vec{v}_1, \dots, \vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_r$ are linearly dependent by the previous proposition. $\square$

Def A system $\vec{v}_1, \dots, \vec{v}_p \in V$ is called a basis if it is linearly independent and spans V. $\square$

Ex 1 In the first example of this section, we showed

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{F}^n$$

was a basis. It is known as the standard basis for $\mathbb{F}^n$. Also recall that $\vec{e}_2, \dots, \vec{e}_n$ do not span $\mathbb{F}^n$ and hence is not a basis.

2 Consider

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{F}^2$$

Then $\vec{v}_1, \vec{v}_2$ is a basis. Indeed, if $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \vec{0}$ then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 \end{pmatrix}$$

So we obtain the system of equations

$$\begin{cases} \alpha_1 + \alpha_2 = 0 \\ \alpha_1 - \alpha_2 = 0 \end{cases}$$

Adding the two equations gives $2\alpha_1 = 0$ or $\alpha_1 = 0$, and the second equation implies $\alpha_2 = \alpha_1 = 0$. Thus $\vec{v}_1, \vec{v}_2$ are linearly independent. To see that they span $\mathbb{F}^2$, fix an arbitrary $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}^2$. We must find scalars β_1, β_2 so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 = \beta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - \beta_2 \end{pmatrix}.$$

Thus we must solve the following system of equations:

$$\begin{cases} \beta_1 + \beta_2 = x \\ \beta_1 - \beta_2 = y \end{cases}$$

Adding the two equations gives $2\beta_1 = x + y$ or $\beta_1 = \frac{x+y}{2}$. Then the second equation gives $\beta_2 = \beta_1 - y = \frac{x-y}{2}$. Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{x+y}{2}\right) \vec{v}_1 + \left(\frac{x-y}{2}\right) \vec{v}_2 \in \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$$

Since $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}^2$ was arbitrary, we see that $\vec{v}_1, \vec{v}_2$ span $\mathbb{F}^2$ and hence is a basis.

Note that this basis is different from $\vec{e}_1, \vec{e}_2$, and so a vector space may have more than one basis.

3. In $\mathbb{P}_n$,

$$p_0(t) = 1, \quad p_1(t) = t, \quad \dots, \quad p_n(t) = t^n$$

is a basis (Exercise check this), which we call the standard basis for $\mathbb{P}_n$. □

The next proposition says the answers to questions 1 and 2 are both "Yes" if and only if we have a basis.

Proposition 1.5 A system of vectors $\vec{v}_1, \dots, \vec{v}_p \in V$ is a basis if and only if for every $\vec{v} \in V$ there exists a unique set of scalars $\alpha_1, \dots, \alpha_p$ such that

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p.$$

Proof ($\Longrightarrow$) Suppose $\vec{v}_1, \dots, \vec{v}_p$ is a basis, and let $\vec{v} \in V$. Since $\vec{v}_1, \dots, \vec{v}_p$ span V then there exist scalars $\alpha_1, \dots, \alpha_p$ such that

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p.$$

Since $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent, Theorem 1.3 implies the scalars are unique.

($\Longleftarrow$) The existence of the scalars for all $\vec{v} \in V$ implies $\vec{v}_1, \dots, \vec{v}_p$ is a spanning system. The uniqueness of the scalars implies, by Theorem 1.3, that it is a linearly independent system. Hence $\vec{v}_1, \dots, \vec{v}_p$ is a basis. □

9/14

Def The unique scalars $\alpha_1, \dots, \alpha_p$ in the previous proposition are called the coordinates of $\vec{v}$ with respect to the basis $\vec{v}_1, \dots, \vec{v}_p$. □

Remark If $\vec{v}_1, \dots, \vec{v}_p$ is a basis in a vector space V, then every $\vec{v}$ is uniquely determined by its coordinates with respect to this basis. Hence if

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p,$$

then we can recover $\vec{v}$ so long as we remember the column vector $(\alpha_1, \dots, \alpha_p)^T \in \mathbb{F}^p$. Also, this isn't just a convenient way to remember $\vec{v}$, it also respects addition and scalar multiplication:

$$\vec{v} + \vec{w} = \left(\sum_{k=1}^{p} \alpha_k \vec{v}_k\right) + \left(\sum_{k=1}^{p} \beta_k \vec{v}_k\right) = \sum_{k=1}^{p} (\alpha_k + \beta_k) \vec{v}_k \longleftrightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_p + \beta_p \end{pmatrix}$$

$$\beta \vec{v} = \beta \sum_{k=1}^{p} \alpha_k \vec{v}_k = \sum_{k=1}^{p} (\beta \alpha_k) \vec{v}_k \longleftrightarrow \beta \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{pmatrix} = \begin{pmatrix} \beta \alpha_1 \\ \vdots \\ \beta \alpha_p \end{pmatrix}$$

This is the power of a basis: it allows us to take a potentially very abstract vector space and replace it with something much more familiar ($\mathbb{F}^p$). □

Proposition 1.6 Any finite spanning system contains a basis.

Proof Suppose $\vec{v}_1, \ldots, \vec{v}_p \in V$ are a spanning system in a vector space V. If it linearly independent, then it is a basis and we are done. Otherwise it is linearly dependent, and so by Proposition 1.4 there exists $1 \le k \le p$ such that

$$\vec{v}_k = \beta_1 \vec{v}_1 + \cdots + \beta_{k-1}\vec{v}_{k-1} + \beta_{k+1}\vec{v}_{k+1} + \cdots + \beta_p \vec{v}_p.$$

Without loss of generality, we may assume $k=p$. Observe that we can use the above formula to rewrite any linear combination of $\vec{v}_1, \ldots, \vec{v}_p$ as a linear combination of $\vec{v}_1, \ldots, \vec{v}_{p-1}$:

$$\begin{aligned}\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p &= \alpha_1 \vec{v}_1 + \cdots + \alpha_{p-1}\vec{v}_{p-1} + \alpha_p \Big(\sum_{k=1}^{p-1} \beta_k \vec{v}_k \Big) \\ &= (\alpha_1 + \beta_1)\vec{v}_1 + \cdots + (\alpha_{p-1} + \beta_{p-1})\vec{v}_{p-1}.\end{aligned}$$

Thus

$$V = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_p\} \subset \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{p-1}\} \subset V,$$

which implies $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{p-1}\} = V$. So $\vec{v}_1, \ldots, \vec{v}_{p-1}$ is a strictly smaller spanning system. If it is linearly independent, then it is a basis and we are done. Otherwise we iterate the above argument until we either obtain a basis or reduce down to a single vector $\vec{v}_1$ which spans V. Since a system consisting of a single vector is always linearly independent (Exercise), even in this case we have obtained a basis. Note that in all cases the resulting basis is a subset of $\{\vec{v}_1, \ldots, \vec{v}_p\}$. □

1.3 Linear Transformations and Matrix-vector multiplication

- A "transformation" T from a set X to a set Y is a rule that for each input $x \in X$ assigns an output $y \in Y$, which we denote $T(x) := y$. We write

$$T : \underset{\text{domain}}{X} \to \underset{\text{target space/codomain}}{Y}$$

synonyms: transform, mapping, map, operation, or function.

- If the sets X, Y have more structure, then so can T.

Def Let V, W be vector spaces (over the same field $\mathbb{F}$). A transformation $T: V \to W$ is called <u>linear</u> if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall\, \vec{u}, \vec{v} \in V$ "for all"
2. $T(\alpha \vec{v}) = \alpha T(\vec{v})$ for all $\vec{v} \in V$ and all scalars $\alpha \in \mathbb{F}$.

Note that we can combine properties 1. and 2. into a single, equivalent statement:

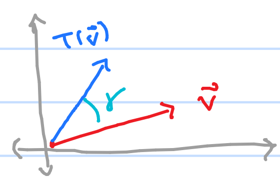
$$T(\alpha \vec{u} + \beta \vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v}) \qquad \forall\, \vec{u}, \vec{v} \in V \text{ and } \forall\, \alpha, \beta \in \mathbb{F}.$$

Ex ① Let $V = \mathbb{P}_n$, $W = \mathbb{P}_{n-1}$ and define $T: \mathbb{P}_n \to \mathbb{P}_{n-1}$ by $T(p) = p'$. That is,

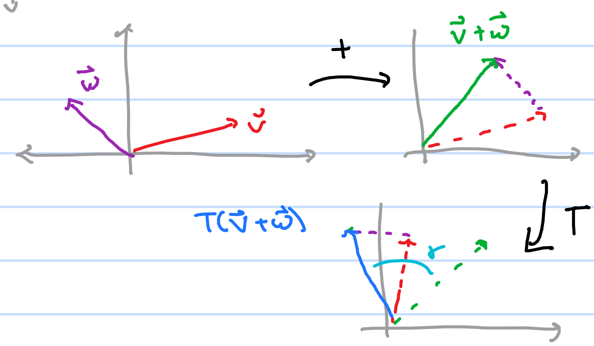
$$T(a_n t^n + \cdots + a_1 t + a_0) = n a_n t^{n-1} + \cdots + a_1$$

Since $(p+q)' = p' + q'$ and $(\alpha p)' = \alpha p'$, T is linear.

② Let $V = W = \mathbb{R}^2$ and fix $\gamma \in [0, 2\pi)$. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by setting $T(\vec{v})$ to be the vector one obtains after rotating the plane counterclockwise by γ radians.



Recall that addition in $\mathbb{R}^2$ is visually equivalent to concatenating vectors and forming a triangle. Since rotating the plane preserves the internal angles of the triangle, we see that T satisfies Property 1. Property 2 is also easily checked, so T is a linear transformation.



③ Let $V = \{ f : \mathbb{R} \to \mathbb{R} \}$ and $W = \mathbb{R}$. Define $T: V \to \mathbb{R}$ by $T(f) = f(2)$. Then for $f, g \in V$ and $\alpha, \beta \in \mathbb{R}$

$$T(\alpha f + \beta g) = (\alpha f + \beta g)(2) = \alpha(f(2)) + \beta(g(2)) = \alpha T(f) + \beta T(g)$$

So T is a linear transformation.

(4) Let $V = W = \mathbb{R}$.

Claim: Any linear transformation $T: \mathbb{R} \to \mathbb{R}$ is given by

$$T(x) = ax \quad \text{where} \quad a = T(1).$$

Indeed, $x \in \mathbb{R}$ is both a vector in V and a scalar in $\mathbb{R}$: $x = x\,1$ (vectors; scalar)

so because T is linear and satisfies Property 2, we have:

$$T(x) = T(x1) = x\,T(1) = xa = ax.$$

as claimed.

Similarly, any linear transformation $T: \mathbb{C} \to \mathbb{C}$ is determined by multiplication by a scalar $a \in \mathbb{C}$. $\square$

Linear Transformations $\mathbb{F}^n \to \mathbb{F}^m$

Let $n, m \in \mathbb{N}$. We will generalize the claim in the last example to higher dimensions and show any linear transformation $\mathbb{F}^n \to \mathbb{F}^m$ is given by multiplication by a matrix (rather than a scalar).

Let $T: \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation.

Claim 1 To compute $T(\vec{x})$ for any $\vec{x} \in \mathbb{F}^n$, it suffices to know $T(\vec{e}_1), \ldots, T(\vec{e}_n)$ for the standard basis $\vec{e}_1, \ldots, \vec{e}_n$ of $\mathbb{F}^n$.

Indeed, suppose $\vec{x} = (x_1, x_2, \ldots, x_n)^T$. Then

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

So using the linearity of T we have:

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \\ &= T(x_1\vec{e}_1) + \cdots + T(x_n\vec{e}_n) \\ &= x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n) \end{aligned}$$

Thus if

$$\vec{a}_1 := T(\vec{e}_1) \quad \vec{a}_2 := T(\vec{e}_2) \quad \cdots \quad \vec{a}_n := T(\vec{e}_n)$$

Then $T(\vec{x})$ is the linear comb. $\sum_{j=1}^{n} x_j \vec{a}_j$. $\square$

Let's examine this further. Define a matrix $A \in M_{m \times n}$ by

$$A = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{pmatrix}$$

Label the entries $A = (a_{ij})_{i=1, j=1}^{m\ \ n}$, so that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad \vec{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Recall that multiplication by matrices is defined by a "row by column" rule: If $A\vec{x}=\vec{y}$ for $A\in M_{m\times n}$, $\vec{x}\in\mathbb{F}^n$, and $\vec{y}\in\mathbb{F}^m$, then the kth entry of $\vec{y}$ is the dot product of the kth row of A and the column vector $\vec{x}$:

$$y_k = (A)_{k1}x_1+(A)_{k2}x_2+\cdots+(A)_{kn}x_n = \sum_{j=1}^{n}(A)_{kj}x_j \qquad k\Big(\ \overline{\ A\ }\ \Big)\Big(\,\vec{x}\,\Big)=\Big(\,\vec{y}\,\Big)k$$

$\boxed{\text{Claim 2}}$ With $T:\mathbb{F}^n\to\mathbb{F}^m$ and A as before, $T(\vec{x})=A\vec{x}$.

Indeed, we have already seen that

$$T(\vec{x}) = x_1\vec{a}_1+x_2\vec{a}_2+\cdots+x_n\vec{a}_n$$

Expanding the vectors:

$$= x_1\begin{pmatrix}a_{11}\\a_{21}\\ \vdots\\ a_{m1}\end{pmatrix}+x_2\begin{pmatrix}a_{12}\\a_{22}\\ \vdots\\ a_{m2}\end{pmatrix}+\cdots+x_n\begin{pmatrix}a_{1n}\\a_{2n}\\ \vdots\\ a_{mn}\end{pmatrix}$$

$$=\begin{pmatrix}x_1a_{11}+x_2a_{12}+\cdots+x_na_{1n}\\ x_1a_{21}+x_2a_{22}+\cdots+x_na_{2n}\\ \vdots\\ x_1a_{m1}+x_2a_{m2}+\cdots+x_na_{mn}\end{pmatrix}$$

$$=\begin{pmatrix}\sum_{j=1}^n a_{1j}x_j\\ \sum_{j=1}^n a_{2j}x_j\\ \vdots\\ \sum_{j=1}^n a_{mj}x_j\end{pmatrix} = A\vec{x}.$$

$\square$

- Thus any linear transformation $T:\mathbb{F}^n\to\mathbb{F}^m$ can be represented as multiplication by by the matrix $A\in M_{m\times n}$ whose columns are $T(\vec{e}_j)$. Pay careful attention to to 'n' vs 'm' in this representation.
- We will denote the matrix A by $[T]$ or even just T if there is no room for confusion, so we can write $T\vec{v}$ for $T(\vec{v})$.

$\boxed{\text{Rem}}$ In claim 1, we did not need to use the standard basis $\vec{e}_1,\dots,\vec{e}_n$. All we needed was to be able to write $\vec{x}$ as a linear comb. of $\vec{e}_1,\dots,\vec{e}_n$ (which is easy). Thus we could have used any basis for $\mathbb{F}^n$ or even any ~~generating~~ spanning set. This is even true for arbitrary vector spaces.

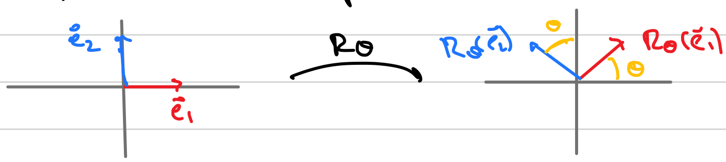
$\boxed{\text{Prop}}$ 1.7 A linear transformation $T:V\to W$ is completely determined by its output on a ~~generating~~ spanning set (in particular by its output on a basis).

Proof Exercise. $\square$

$\boxed{\text{Ex}}$ For $\theta\in[0,2\pi)$, let $R_\theta:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation that rotates the plane counterclockwise by θ radians. Let's compute $[R_\theta]$. Recall from above that

$$[R_\theta] = \left(R_\theta(\vec{e}_1) \quad R_\theta(\vec{e}_2) \right)$$

So it suffices to compute these two vectors. Observe



So and

$$R_\theta(\vec{e}_1) = (\cos\theta, \sin\theta)^T$$

$$R_\theta(\vec{e}_2) = (\cos(\theta + \tfrac{\pi}{2}), \sin(\theta + \tfrac{\pi}{2}))^T = (-\sin\theta, \cos\theta)^T$$

Hence

$$[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

□

1.4 Linear transformations as a vector space

Let us explore what operations we can perform on linear transformations themselves.

Suppose $S, T: V \to W$ are linear transformations. Then for any $\vec{v} \in V$ we can add $S(\vec{v}) + T(\vec{v})$. Denote $(S+T)(\vec{v}) = S(\vec{v}) + T(\vec{v})$, then this defines a new trans. $S+T: V \to W$. Is it linear? For $\vec{v}, \vec{w} \in V$ and scalars α, β we have

$$
\begin{aligned}
(S+T)(\alpha\vec{v}+\beta\vec{w}) &= S(\alpha\vec{v}+\beta\vec{w}) + T(\alpha\vec{v}+\beta\vec{w}) \\
&= \alpha S(\vec{v}) + \beta S(\vec{w}) + \alpha T(\vec{v}) + \beta T(\vec{w}) \\
&= \alpha[S(\vec{v})+T(\vec{v})] + \beta[S(\vec{w})+T(\vec{w})] \\
&= \alpha(S+T)(\vec{v}) + \beta(S+T)(\vec{w})
\end{aligned}
$$

So $S+T$ is also linear.

For $T: V \to W$ a lin. trans. and α a scalar define a trans. $\alpha T: V \to W$ by

$$(\alpha T)(\vec{v}) = \alpha T(\vec{v}) \qquad \vec{v} \in V.$$

One can show αT is also linear (Exercise).

Let $L(V, W)$ denote the collection of linear transformations from a vector space V to a vector space W. We have shown above that it admits operations of addition and scalar multiplication. Moreover, one can check that these axioms satisfy the vector space axioms. For example:

- Zero vector: let $\vec{0}_W$ be the zero vector in W. Then $O: V \to W$ defined by

$$O(\vec{v}) = \vec{0}_W \qquad \vec{v} \in V$$

is the zero "vector" in $L(V, W)$. Indeed, it is linear:

$$O(\alpha\vec{v}+\beta\vec{w}) = \vec{0}_W = \vec{0}_W + \vec{0}_W \overset{HW1}{=} \alpha\vec{0}_W + \beta\vec{0}_W = \alpha O(\vec{v}) + \beta O(\vec{w}).$$

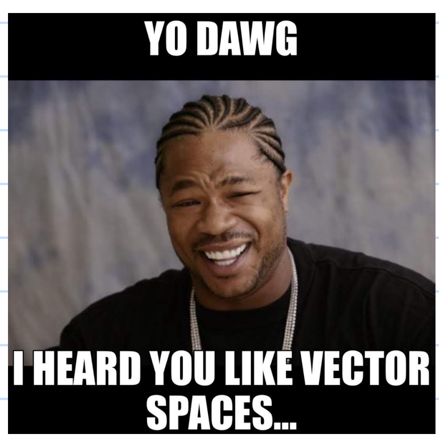
So $O \in L(V, W)$. And for any $T \in L(V, W)$ we have $T + O = T$ since

$$(T+O)(\vec{v}) = T(\vec{v}) + O(\vec{v}) = T(\vec{v}) + \vec{0}_W = T(\vec{v})$$

for all $\vec{v} \in V$.

(Exercise: check the remaining axioms).

Thus $L(V, W)$ is itself a vector space.



$\boxed{\text{Ex}}$ For $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$, $L(\mathbb{F}^n, \mathbb{F}^m)$ is a vector space. On the other hand, we know every $T \in L(\mathbb{F}^n, \mathbb{F}^m)$ can be represented as matrix multiplication by $[T]$. Moreover, the operations on $L(\mathbb{F}^n, \mathbb{F}^m)$ match those on $M_{m \times n}$:

$$[S+T] = [S] + [T]$$
$$[\alpha T] = \alpha [T]$$

So $L(\mathbb{F}^n, \mathbb{F}^m) = M_{m \times n}$. $\square$

$\boxed{\text{Rem}}$ What about the other operation we have for $M_{m \times n}$: multiplication? It turns out this corresponds to composition of linear trans, as we will see in the next section.

1.5 Composition of linear transformations (and matrix multiplication)

For two matrices A, B recall how their product is defined:

Then entry $(AB)_{jk}$ is given by the dot product of the jth row of A and the kth column of B:

$$(AB)_{jk} = \sum_{\ell} (A)_{j\ell} (B)_{\ell k} \qquad j\begin{pmatrix} \quad A \quad \end{pmatrix} \overset{k}{\begin{pmatrix} B \end{pmatrix}} = \begin{pmatrix} \overset{jk}{AB} \end{pmatrix}$$

Warning AB only makes sense if # columns of A = # rows of B.

That is

$$\underset{A}{M_{m\times \boxed{n}}} \qquad \underset{B}{M_{\boxed{n}\times r}}$$

Composition of Linear Transformations

Suppose $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^r \to \mathbb{F}^n$. Then we define $T_1 \circ T_2 : \mathbb{F}^r \to \mathbb{F}^m$ by

$$(T_1 \circ T_2)(\vec{v}) = \underbrace{T_1(\underbrace{T_2(\vec{v})}_{\in \mathbb{F}^n})}_{\text{note order}} \qquad \vec{v} \in \mathbb{F}^r$$

Claim $[T_1 \circ T_2] = [T_1][T_2]$ ← matrix product

Proof: Denote $T = T_1 \circ T_2$ and let $A = [T_1]$, $B = [T_2]$. Recall that $[T]$ is simply the matrix whose columns are given by $T(\vec{e}_1), \ldots, T(\vec{e}_r)$. For $j = 1, \ldots, r$ we have

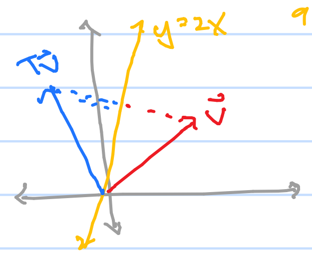
$$T(\vec{e}_j) = T_1 \circ T_2(\vec{e}_j) = T_1(T_2(\vec{e}_j)) = T_1(B\vec{e}_j) = A(B\vec{e}_j)$$
$$= AB\vec{e}_j$$

So the jth column of $[T]$ is $AB\vec{e}_j$, but this is precisely the jth column of AB. So $[T] = AB$ since they have the same columns. $\square$

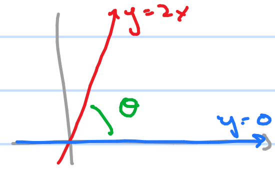
9/21

Ex Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that sends $\vec{v} \in \mathbb{R}^2$ to its reflection over the line $y = 2x$. Let's compute $[T]$. It suffices to compute $T(\vec{e}_1), T(\vec{e}_2)$ for

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



but this is hard. Instead, we note that if the line were $y = 0$, reflecting would be much easier to compute: $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$.



But remember that rotation is a linear transformation, so we can first rotate so $y=2x$ becomes $y=0$, reflect, and then rotate back.

Let θ be the angle between $y=2x$ and $y=0$.

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by θ radians counterclockwise.

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over $y=0$.

Then

$$T = R_\theta \circ S \circ R_{-\theta}$$

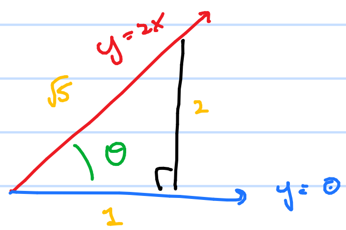
So $[T] = [R_\theta][S][R_{-\theta}]$.

First we compute $[S]$:

$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} \quad \Rightarrow \quad [S] = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

Recall: $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$

Note that



so $\cos\theta = \frac{1}{\sqrt5} = \frac{\sqrt5}{5}$

$\sin\theta = \frac{2}{\sqrt5} = \frac{2\sqrt5}{5}$

So $R_\theta = \frac{\sqrt5}{5}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}$ and $R_{-\theta} = \begin{pmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} = \frac{\sqrt5}{5}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}$

Hence

$$[T] = \frac{1}{5}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-3 & 4\\4 & 3\end{pmatrix} \qquad \square$$

Properties of matrix multiplication

Matrix multiplication satisfies the following properties:

1. Associativity: $A(BC) = (AB)C$, provided that either the left or right side is well-defined; we therefore just write ABC.
2. Distributivity: $A(B+C) = AB + AC$
   $(A+B)C = AC + BC$
3. Commutativity with scalar multiplication: $A(\alpha B) = \alpha(AB) = (\alpha A)B$

Exercise: prove these properties.

These are just the usual multiplication properties that numbers satisfy, except multiplication is not commutative: generally $AB \neq BA$.

Ex
- $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}1&2\\0&0\end{pmatrix}$ while $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\3&0\end{pmatrix}$.
- $\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}1&2\\0&0\\0&0\end{pmatrix}$ while $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}$ DNE $\square$

Recall that the transpose A^T of a matrix A is found by turning the rows of A into the columns of A^T (or vice-versa): $(A^T)_{jk} = (A)_{kj}$.

Prop 1.8 Let $A \in M_{m\times n}$ and $B \in M_{n\times r}$ be matrices. Then

$$(AB)^T = B^T A^T$$

Proof We simply appeal to the definition of matrix multiplication and the transpose:

$$((AB)^T)_{jk} = (AB)_{kj} = \sum_{\ell=1}^{n} (A)_{k\ell}(B)_{\ell j}$$

while

$$(B^T A^T)_{jk} = \sum_{\ell=1}^{n} (B^T)_{j\ell}(A^T)_{\ell k} = \sum_{\ell=1}^{n} (B)_{\ell j}(A)_{k\ell} = \sum_{\ell=1}^{n} (A)_{k\ell}(B)_{\ell j}$$

$=\checkmark$

So the entries of $(AB)^T$ agree with the entries of $B^T A^T$, which means $(AB)^T = B^T A^T$. $\square$

The Trace

Def For $A \in M_{n\times n}$ (a square matrix) its trace is the scaler

$$\mathrm{tr}(A) := (A)_{11} + (A)_{22} + \cdots + (A)_{nn} = \sum_{i=1}^{n} (A)_{ii}$$

- Observe that the trace defines a transformation $\mathrm{tr}: M_{n\times n} \longrightarrow \mathbb{F}$. It is in fact linear: for $A, B \in M_{n\times n}$ and scalars $\alpha, \beta \in \mathbb{F}$

$$\mathrm{tr}(\alpha A + \beta B) = \sum_{i=1}^{n} (\alpha A + \beta B)_{ii} = \sum_{i=1}^{n} (\alpha A)_{ii} + (\beta B)_{ii}$$

$$= \sum_{i=1}^{n} \alpha (A)_{ii} + \beta (B)_{ii} = \alpha \sum_{i=1}^{n} (A)_{ii} + \beta \sum_{i=1}^{n} (B)_{ii} = \alpha\,\mathrm{tr}(A) + \beta\,\mathrm{tr}(B).$$

Thm 1.9 Let $A \in M_{m\times n}$ and $B \in M_{n\times m}$ be matrices. Then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

Proof This is Exercise 6 on Homework 3. $\square$

1.6 Invertible Transformations: Isomorphisms

- Recall that if $T: \mathbb{R} \to \mathbb{R}$ is linear, then for all $x \in \mathbb{R}$ $T(x) = ax$ for some $a \in \mathbb{R}$. Also recall that if $a \neq 0$ the "inverse" (reciprocal) of a is $\frac{1}{a}$. Define $S: \mathbb{R} \to \mathbb{R}$ by $S(x) = (\frac{1}{a})x$. Let's compute $T \circ S$ and $S \circ T$:

$$T \circ S(x) = T(\tfrac{1}{a}x) = a(\tfrac{1}{a}x) = x$$

$$S \circ T(x) = S(ax) = \tfrac{1}{a}(ax) = x$$

So $T \circ S$ and $S \circ T$ both equal the transformation $I: \mathbb{R} \to \mathbb{R}$ defined by $I(x) = x$. We call I the identity transformation, and can easily see it is linear.

- More generally we have:

Def For a vector space V, the identity (linear) transformation $I_V: V \to V$ is defined by $I_V(\vec{x}) = \vec{x}$. We will often just write I for I_V

- Note that the domain and target space of I are the same.

Ex Let $I: \mathbb{F}^n \to \mathbb{F}^n$ be the identity transformation. Let's compute $[I]$:

$$I(\vec{e}_1) = \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad I(\vec{e}_2) = \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, I(\vec{e}_n) = \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Thus

$$[I] = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

We will denote this matrix by I_n. □

- Let $T: V \to V$ be a linear transformation. Observe that

$$T \circ I(\vec{v}) = T(I(\vec{v})) = T(\vec{v})$$

$$I \circ T(\vec{v}) = I(T(\vec{v})) = T(\vec{v})$$

So $T \circ I = T = I \circ T$. Compare this to $t \cdot 1 = t = 1 \cdot t$ for $t \in \mathbb{R}$.
A special case of this fact is that for $A \in M_{n \times n}$

$$A I_n = A = I_n A.$$

Exercise: use matrix multiplication to show the above equalities.

- Recall that in the first example above, we had $S \circ T = T \circ S = I$. More generally, we have:

Def Let $A: V \to W$ be a linear transformation. We say that A is

left invertible if there exists a linear transformation $B: W \to V$ such that

$$B \circ A = I,$$

where here $I = I_V$.

We say A is **right invertible** if there exists a linear transformation $C: W \to V$ such that

$$A \circ C = I,$$

where here $I = I_W$

We call B and C the left and right inverses of A, respectively.

$\boxed{\text{Def}}$ A linear transformation $A: V \to W$ is **invertible** if it is both left and right invertible

$\boxed{\text{Ex}}$ Consider the linear transformation $A: \mathbb{P}_3 \to \mathbb{P}_2$ defined by $A(p(x)) = p'(x)$.

$\boxed{\text{Claim 1}}$ A is right invertible

$\boxed{\text{Proof}}$ We must find a lin. trans. $B: \mathbb{P}_2 \to \mathbb{P}_3$ so that $A \circ B = I_{\mathbb{P}_2}$. That is, for $p(x) \in \mathbb{P}_2$, we have

$$A \circ B(p(x)) = I(p(x))$$
$$A(B(p(x)) = p(x)$$

So $q(x) := B(p(x)) \in \mathbb{P}_3$ must be a polynomial s.t. $A(q(x)) = q'(x) = p(x)$. Thus $q(x)$ should be an anti-derivative of $p(x)$. If $p(x) = a_2x^2 + a_1x + a_0$, then

$$B(p(x)) = \tfrac{1}{3}a_2x^3 + \tfrac{1}{2}a_1x^2 + a_0x + C$$

for some scalar C. However, since we want B to be linear, we must choose $C = 0$. Indeed,

$$B(2x) = x^2 + C$$
$$\|$$
$$B(x+x) = B(x) + B(x) = \tfrac{1}{2}x^2 + C + \tfrac{1}{2}x^2 + C = x^2 + 2C$$

So $2C = C \Rightarrow C = 0$. So define B by

$$B(a_2x^2 + a_1x + a_0) = \tfrac{1}{3}a_2x^3 + \tfrac{1}{2}a_1x^2 + a_0x$$

(Exercise: Check that B satisfies the full def. of linear.

It is easy to see that $A(B(p(x)) = p(x)$ for all $p(x) \in \mathbb{P}_2$, so A is right invertible with right inverse B. $\square$

$\boxed{\text{Claim 2}}$ A is not left invertible.

$\boxed{\text{Proof}}$ We'll do a proof by contradiction. Suppose, towards a a contradiction, that $C: \mathbb{P}_2 \to \mathbb{P}_3$ was a left inverse for A:

$$C \circ A = I_{\mathbb{P}_3}$$

Then $C(A(p(x)) = p(x)$ for every $p(x) \in \mathbb{P}_3$. Consider $p(x) = 1$. Then $A(p(x)) = (1)' = 0$. Since C is linear, we must have

$$C(A(p(x))) = C(0) = 0 \neq p(x)$$

a contradiction. Thus the left inverse of A must not exist, and so A is not left invertible. This further implies A is not invertible. $\square$ $\square$

- The above example implies there are transformations that are right invertible, but not left invertible. Similarly, there are trans. that are left but not right invertible. transformations (e.g. B)

9/23

Thm 1.10 If a linear transformation $A: V \to W$ is invertible, then its left and right inverses B and C are unique and satisfy $B = C$

Proof By definition of the left and right inverses, we have

$$B \circ A = I_V \quad \text{and} \quad A \circ C = I_W$$

So

$$(B \circ A) \circ C = I_V \circ C$$
$$B \circ (A \circ C) = C$$
$$B \circ (I_W) = C$$
$$B = C$$

Now, if $B_1 : W \to V$ is another left inverse of A, then repeating the argument with B_1 instead of B gives $B_1 = C$. But since $C = B \Rightarrow B_1 = B$. Thus the left inverse is unique. A similar proof shows the right inverse is unique. $\square$

- This theorem can be used to simplify our proof of Claim 2 in the previous example. Rather than showing a left inverse cannot exist, we just have to show it cannot be B from Claim 1.

Corollary 1.11 A linear transformation $A: V \to W$ is invertible if and only if there exists a unique linear transformation, denoted A^{-1}, such that $A^{-1}: W \to V$ and

$$A^{-1}A = I_V \quad \text{and} \quad AA^{-1} = I_W.$$

A^{-1} is called the inverse of A.

Matrix Inverses

Def A matrix $\in M_{m \times n}$ is invertible (resp. left invertible, right invertible) if the linear transformation

$$F^n \ni \vec{x} \mapsto A\vec{x} \in F^m$$

is invertible (resp. left invertible, right invertible).

The above theorem says that if $A \in M_{m \times n}$ is invertible, then there is a unique matrix A^{-1} satisfying

$AA^{-1} = I$ and $A^{-1}A = I$. We, of course, call A^{-1} the inverse of A.

$\boxed{\text{EX}}$ ① The identity matrix $I_n \in M_{n\times n}$ is invertible with $(I_n)^{-1} = I_n$. (Compare this to the reciprocal of 1 being 1).

② For $\theta \in [0, 2\pi)$ the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

has inverse $(R_\theta)^{-1} = R_{-\theta}$. This clear from how the rotation is defined.
Exercise: verify using matrix multiplication: $R_\theta R_{-\theta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $R_{-\theta}R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

③ For $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $B = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$ is a left inverse:

$$BA = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2} + \tfrac{1}{2} \end{pmatrix} = (1) = I.$$

But it isn't a right inverse since AB ~~is not defined~~ $= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \neq I$. So A is not right invertible

Thm 1.12 If $A \in M_{m\times n}$ is invertible, then A is square (i.e. $n = m$).

Proof Since $A^{-1} \in M_{n\times m}$, we have

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_m$$

So using Exercise #6 on HW 3 we have:

$$n = \operatorname{Tr}(I_n) = \operatorname{Tr}(A^{-1}A) = \operatorname{Tr}(AA^{-1}) = \operatorname{Tr}(I_m) = m. \qquad \square$$

Properties of the inverse transformation

Thm 1.13 If $A : V \to W$ and $B : U \to V$ are invertible linear transformations, then $A \circ B : V \to U$ is invertible with

$$(A \circ B)^{-1} = B^{-1} \circ A^{-1}$$

Proof We compute

$$(A\circ B)\circ(B^{-1}\circ A^{-1}) = A\circ(B\circ B^{-1})\circ A^{-1} = A\circ I\circ A^{-1} = A\circ A^{-1} = I$$

and

$$(B^{-1}\circ A^{-1})\circ(A\circ B) = B^{-1}\circ(A^{-1}\circ A)\circ B = B^{-1}\circ I\circ B = B^{-1}\circ B = I \qquad \square$$

Rem Be careful: if $A\circ B$ is invertible this does not imply A and B are invertible.
~~In fact, in the above proof, we really only used that B is right inv and A is left inv~~

Thm 1.14 If A is an invertible matrix, then A^T is invertible with $(A^T)^{-1} = (A^{-1})^T$!

Proof Recall $(AB)^T = B^T A^T$. So we have:

$$(A^{-1})^T A^T = (AA^{-1})^T = (I)^T = I$$

and

$$A^T (A^{-1})^T = (A^{-1}A)^T = (I)^T = I. \qquad \square$$

Thm 1.15 If $A: V \to W$ is invertible, then so is A^{-1} with $(A^{-1})^{-1} = A$.

Proof The same equations that show A^{-1} is the inverse of A also show $A = (A^{-1})^{-1}$. $\square$

Isomorphism and Isomorphic Spaces

Def Given two vector spaces V and W (over the same field $\mathbb{F}$), we say V and W are isomorphic, and write $V \cong W$, if there exists an invertible linear transformation $T: V \to W$. We call T an isomorphism.

- When V and W are isomorphic, this means they are effectively the same vector space, just with another name. Let's see some evidence of this:

Thm 1.16 Let $T: V \to W$ be an isomorphism, and let $\vec{v}_1, \ldots, \vec{v}_n \in V$ be a generating (resp. linearly independent) system. Then $T(\vec{v}_1), \ldots, T(\vec{v}_n) \in W$ is a generating (resp. linearly independent) system. In particular, if $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for V, then $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ is a basis for W.

Proof Homework 4 $\square$

Thm 1.17 Let $T: V \to W$ be a linear transformation, and let $\vec{v}_1, \ldots, \vec{v}_n \in V$ be a basis. If $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ is a basis for W, then T is an isomorphism.

Proof Denote

$$\vec{w}_1 := T(\vec{v}_1), \ldots, \vec{w}_n := T(\vec{v}_n)$$

Recall that any linear transformation is defined by its outputs on a basis. Thus we can define a lin. trans. $S: W \to V$ by $S(\vec{w}_j) := \vec{v}_j$. It follows that

$$T \circ S(\vec{w}_j) = T(\vec{v}_j) = \vec{w}_j = I_W(\vec{w}_j)$$
$$S \circ T(\vec{v}_j) = S(\vec{w}_j) = \vec{v}_j = I_V(\vec{v}_j)$$

Since $T \circ S$ and $S \circ T$ are determined by their outputs on the bases $\vec{w}_1, \ldots, \vec{w}_n$ and $\vec{v}_1, \ldots, \vec{v}_n$, respectively, it follows that $T \circ S = I_W$ and $S \circ T = I_V$. That is, $S = T^{-1}$ and so T is an isomorphism. $\square$

Cor 1.18 $A \in M_{n\times n}$ is invertible if and only if its columns form a basis for $\mathbb{F}^n$.

Proof The columns of A are the vectors $A(\vec{e}_1), \dots, A(\vec{e}_n)$. So the previous two theorems yield ($\Rightarrow$) and ($\Leftarrow$), respectively. □

Ex ① $\mathbb{P}_n \cong \mathbb{F}^{n+1}$. Define $T: \mathbb{F}^{n+1} \to \mathbb{P}_n$ by

$$T(\vec{e}_1) = 1, \quad T(\vec{e}_2) = x, \quad \dots, \quad T(\vec{e}_{n+1}) = x^n$$

Since $1, x, \dots, x^n$ is a basis for $\mathbb{P}_n$, the previous theorem implies T is an iso.

② Let V be any real vector space with basis $\vec{v}_1, \dots, \vec{v}_n \in V$. Then $V \cong \mathbb{R}^n$ by the iso

$$T(\vec{e}_1) = \vec{v}_1, \dots, T(\vec{e}_n) = \vec{v}_n.$$

Similarly, if V is a complex vector space with basis $\vec{v}_1, \dots, \vec{v}_n$, then $V \cong \mathbb{C}^n$.

③ $M_{2\times 3} \cong \mathbb{F}^{6}$ $(= 2\cdot 3)$ since $\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}$ is a basis.

Rem In general, $M_{m\times n} \cong \mathbb{F}^{m\cdot n}$ and an isomorphism is defined by:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ a_{12} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

i.e. stack the columns of A on top of each other

However, this forgets some important information about $M_{m\times n}$: mult. and transpose.

Thm 1.19 Let V be a vector space, and suppose $\vec{v}_1, \dots, \vec{v}_n$ and $\vec{w}_1, \dots, \vec{w}_m$ are both bases for V. Then $n = m$.

Proof By the above example, we have $V \cong \mathbb{F}^n$, say with iso. $T: V \to \mathbb{F}^n$, and $V \cong \mathbb{F}^m$ say with iso. $S: V \to \mathbb{F}^m$. Then $S \circ T^{-1}: \mathbb{F}^n \to \mathbb{F}^m$ is an isomorphism, and therefore $[S \circ T^{-1}] \in M_{m\times n}$ is invertible. Since only square matrices are invertible, we have $n = m$. □

- So any two finite bases of a vector space have the same size.

Def Let V be a vector space and let $\vec{v}_1, \dots, \vec{v}_n$ be any basis for V. The dimension of V, denoted $\dim(V)$, is the number n.

Invertibility and Solving Equations

1.20

$\boxed{\text{Thm}}$ Let $A: X \to Y$ be a linear transformation. Then A is invertible if and only if for every $\vec{b} \in Y$ the equation

$$A(\vec{x}) = \vec{b}$$

has a unique solution.

Proof ($\Longrightarrow$) Suppose A is invertible. For $\vec{b} \in Y$, $\vec{x} := A^{-1}(\vec{b})$ solves the equation. Moreover, if $\vec{x}_1$ is another solution then we have:

$$A(\vec{x}_1) = \vec{b}$$
$$A^{-1}(A(\vec{x}_1)) = A^{-1}(\vec{b})$$
$$\vec{x}_1 = \vec{x}.$$

So the solution is unique.

($\Longleftarrow$) Suppose for any $\vec{b} \in Y$, $A(\vec{x}) = \vec{b}$ has a unique solution. Define $B: Y \to X$ by letting $B(\vec{y}) \in X$ be the unique solution of $A(\vec{x}) = \vec{y}$.

We claim B is the inverse of A. First, let us verify that it is linear. For $\vec{y}_1, \vec{y}_2 \in Y$ let $\vec{x}_1 = B(\vec{y}_1)$ and $\vec{x}_2 = B(\vec{y}_2)$. Then by definition of B we have:

$$A(\vec{x}_1) = \vec{y}_1$$
$$A(\vec{x}_2) = \vec{y}_2.$$

Now, for any scalars α, β we have by linearity of A that

$$A(\alpha \vec{x}_1 + \beta \vec{x}_2) = \alpha A(\vec{x}_1) + \beta A(\vec{x}_2) = \alpha \vec{y}_1 + \beta \vec{y}_2.$$

So $\alpha \vec{x}_1 + \beta \vec{x}_2$ must be the unique solution to

$$A(\vec{x}) = \alpha \vec{y}_1 + \beta \vec{y}_2.$$

Hence

$$B(\alpha \vec{y}_1 + \beta \vec{y}_2) = \alpha \vec{x}_1 + \beta \vec{x}_2 = \alpha B(\vec{y}_1) + \beta B(\vec{y}_2).$$

And so B is linear.

Finally, we check $A \circ B = I$ and $B \circ A = I$. Let $\vec{x} \in A$ and set $\vec{y} := A(\vec{x})$. Then by def. of B we have

$$\vec{x} = B(\vec{y}) = B(A(\vec{x})) = B \circ A(\vec{x}).$$

Similarly, if $\vec{y} \in Y$, then $\vec{x} := B(\vec{y})$ solves $A(\vec{x}) = \vec{y}$. Hence

$$\vec{y} = A(\vec{x}) = A(B(\vec{y})) = A \circ B(\vec{y}).$$

Thus $B = A^{-1}$ as claimed. $\square$

1.7 Subspaces

Def A subspace of a vector space V is a subset $V_0 \subset V$ satisfying:

1. $\vec{0} \in V_0$
2. For every $\vec{v}, \vec{w} \in V_0$, $\vec{v} + \vec{w} \in V_0$ (V_0 is closed under addition)
3. For every $\vec{v} \in V_0$ and every scalar α, $\alpha\vec{v} \in V_0$ (V_0 is closed under scalar multiplication)

- A subspace $V_0 \subset V$ is in particular a vector space in its own right. Indeed, all the axioms are satisfied because they are satisfied for V. Thus subspaces can very easily give us lots of new examples of vector spaces, since we only need to check 1-3 above, rather than all of the vector space axioms.

EX ① For a vector space V, $V_0 = \{\vec{0}\}$ is a subspace called the trivial subspace. It is the smallest subspace in any vector space. ($\phi \subset V$ is not a subspace because it fails 1.)
$V \subset V$ is also a subspace (the biggest subspace).

② Let $T: V \to W$ be a linear transformation.

- $Null(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$ is a subspace of V called the null space of T. (also called the kernel of T and denoted $Ker(T)$).

- $Ran(T) = \{\vec{w} \in W : \exists \vec{v} \in V \text{ such that } T(\vec{v}) = \vec{w}\}$ is a subspace of W called the range of T.

③ Let $\vec{v}_1, \ldots, \vec{v}_r \in V$. The span of $\vec{v}_1, \ldots, \vec{v}_n$ is the set

$$span\{\vec{v}_1, \ldots, \vec{v}_n\} = \{\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n : \alpha_1, \ldots, \alpha_n \text{ are scalars}\},$$

and it is a subspace of V.