Exercises:

1. Does the following system of vectors form a basis in $\mathbb{R}^3$? Justify your answer.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

2. Let $\mathbb{P}_2(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most 2. Show that the vectors

$$p_0(x) = 1 \qquad p_1(x) = x \qquad p_2(x) = \frac{1}{3}(2x^2 - 1)$$

form a basis in $\mathbb{P}_2(\mathbb{R})$.

3. The following are True/False. Prove the True statements and provide counterexamples for the False statements.
   (a) Any set containing a zero vector is linearly dependent.
   (b) A basis must contain $\mathbf{0}$.
   (c) Subsets of linearly dependent sets are linearly dependent.
   (d) Subsets of linearly independent sets are linearly independent.
   (e) If $\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$ for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$, then all the scalars $\alpha_1, \ldots, \alpha_n$ are zero.

4. We say a matrix A is **symmetric** if $A^T = A$. Find a basis for the space of symmetric 2×2 matrices (and prove it is in fact a basis). Make note of the number of elements in your basis.

5. Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ be a system of vectors that is linearly independent but **not** spanning. Show that one can find a vector $\mathbf{v}_{p+1} \in V$ so that the larger system $\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}$ is still linearly independent.

6. Suppose $\mathbf{v}_1, \mathbf{v}_2$ form a basis in a vector space V. Define $\mathbf{w}_1 := \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 := \mathbf{v}_1 - \mathbf{v}_2$. Prove that $\mathbf{w}_1, \mathbf{w}_2$ is also a basis in V.

7. In each of the following, prove whether or not the given transformation is a linear transformation.
   (a) $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left((x,y,z)^T\right) = (x+3y, -z)^T$.
   (b) $T\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $T\left((x,y,z)^T\right) = x + 4$.
   (c) Let V be the space of functions of the form $f\colon \mathbb{R} \to \mathbb{R}$ with the usual addition and scalar multiplication. Define $T\colon V \to V$ by $T(f) = f^2$. That is,

$$[T(f)](x) = f(x)^2 \qquad x \in \mathbb{R}.$$

Solutions:

1. No, the system is linearly dependent: $3\mathbf{v}_1 + 2\mathbf{v}_2 + (-1)\mathbf{v}_3 = \mathbf{0}$. □

2. Note that p_0, p_1, p_2 is *almost* the standard basis for $\mathbb{P}_3(\mathbb{R})$, with the exception of p_2. However, we do have that

$$\frac{3}{2}p_2(x) + \frac{1}{2}p_0(x) = \frac{1}{2}(2x^2 - 1) + \frac{1}{2} = x^2.$$

Thus for arbitrary $p(x) = a_2x^2 + a_1x + a_0$ we have

$$\begin{aligned} p(x) &= a_2\left(\frac{3}{2}p_2(x) + \frac{1}{2}p_0(x)\right) + a_1p_1(x) + a_0p_0(x) \\ &= \left(a_0 + \frac{1}{2}a_2\right)p_0(x) + a_1p_1(x) + \frac{3}{2}a_2p_2(x). \qquad (1) \end{aligned}$$

So every vector in $\mathbb{P}_2(\mathbb{R})$ admits a representation as a linear combination of p_0, p_1, p_2. It remains to show that this linear combination is unique. Suppose $p(x) = b_0 p_0(x) + b_1 p_1(x) + b_2 p_2(x)$. Plugging in the formulas for p_0, p_1, p_2 we obtain

$$p(x) = \left(b_0 - \frac{1}{3}b_2\right) + b_1 x + \frac{2}{3}b_2 x^2.$$

This implies

$$\begin{aligned} b_0 - \frac{1}{3}b_2 &= a_0 \\ b_1 &= a_1 \\ \frac{2}{3}b_2 &= a_2. \end{aligned}$$

Solving these equations for b_0, b_1, b_2 yields $b_0 = a_0 + \frac{1}{2}a_2$, $b_1 = a_1$, and $b_2 = \frac{3}{2}a_2$. Hence the linear combination in (1) is unique. □

3. (a) True. Consider the linear combination where the coefficient of $\mathbf{0}$ is one and the rest of the coefficients are zero, which clearly yields $\mathbf{0}$. Since there is at least one non-zero coefficient, this is a non-trivial linear combination, and so the system is linearly dependent. □

   (b) False. The standard basis for $\mathbb{R}^3$ does not contain $\mathbf{0}$.

   (c) False. Let $\mathbf{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ be any vector and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis for $\mathbb{R}^3$. Then the system $\mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is linearly dependent since

   $$\mathbf{v} + (-v_1)\mathbf{e}_1 + (-v_2)\mathbf{e}_2 + (-v_3)\mathbf{e}_3 = \mathbf{0}.$$

   But on the other hand, the subset $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is linearly independent since it is a basis.

   (d) True. Suppose the system $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ is linearly independent. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\} \subset \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be any subset. Suppose

   $$\sum_{i=1}^{q} \alpha_i \mathbf{w}_i = \mathbf{0}.$$

   We can view the left-hand side as a linear combination of the full system $v_1, \ldots, v_p$, where the coefficients of the missing vectors are all zero:

   $$\sum_{i=1}^{q} \alpha_i \mathbf{w}_i = \sum_{i=1}^{q} \alpha_i \mathbf{w}_i + \sum_{\mathbf{v}_j \notin \{\mathbf{w}_1, \ldots, \mathbf{w}_q\}} 0\mathbf{v}_j.$$

   Since the original system is linearly independent, it must be that all the scalar coefficients are zero. In particular, $\alpha_1 = \cdots = \alpha_q = 0$ and so the system $\mathbf{w}_1, \ldots, \mathbf{w}_q$ is linearly independent. □

   (e) False. Let $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$ in any vector space. Then $(1)\mathbf{v}_1 + (1)\mathbf{v}_2 = \mathbf{0}$.

4. First note that if $A = A^T$ is a symmetric 2×2 matrix, then

   $$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

   for scalars a, b, c. We then have

   $$A = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

   and the matrices

   $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are symmetric. So these three matrices form a spanning system. The system is also linearly independent: if

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

for scalars $\alpha_1, \alpha_2, \alpha_3$ then

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus the system is linearly independent and consequently a basis. Finally, we note that the basis contains three vectors. □

5. Since the system is **not** spanning, there must be at least one vector $\mathbf{v} \in V$ which does **not** admit a representation as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$. Fix any such vector and denote it by $\mathbf{v}_{p+1}$. We claim the system $\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ is linearly independent. Indeed, suppose

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p + \alpha_{p+1} \mathbf{v}_{p+1} = \mathbf{0}. \tag{2}$$

We must show $\alpha_1, \dots, \alpha_p, \alpha_{p+1}$ are all zero.

First suppose $\alpha_{p+1} \neq 0$. This means we can divide by α_{p+1} and so we can solve (2) for $\mathbf{v}_{p+1}$:

$$\mathbf{v}_{p+1} = \frac{\alpha_1}{\alpha_{p+1}} \mathbf{v}_1 + \cdots + \frac{\alpha_p}{\alpha_{p+1}} \mathbf{v}_p.$$

But this contradicts $\mathbf{v}_{p+1}$ not admitting a representation as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$. So it must be that $\alpha_{p+1} = 0$.

If $\alpha_{p+1} = 0$, then (2) becomes

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p = \mathbf{0}.$$

Since $\mathbf{v}_1, \dots, \mathbf{v}_p$ is a linearly independent system it follows that the rest of the coefficients must be zero. Thus the system is linearly independent. □

6. By a theorem from lecture, it suffices to show that $\mathbf{w}_1, \mathbf{w}_2$ is spanning and linearly independent.

We first check it is spanning. Observe that

$$\begin{matrix} \mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2 \end{matrix} \iff \begin{matrix} \mathbf{v}_1 = \frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2 \\ \mathbf{v}_2 = \frac{1}{2}\mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2 \end{matrix}.$$

Let $\mathbf{v} \in V$ be an arbitrary vector. Since $\mathbf{v}_1, \mathbf{v}_2$ is a basis, we know there exists scalars α_1, α_2 so that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$. Using the above formulas for the $\mathbf{v}_i$ in terms of the $\mathbf{w}_j$, this becomes:

$$\mathbf{v} = \alpha_1 (\frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2) + \alpha_2 (\frac{1}{2}\mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2) = \frac{\alpha_1 + \alpha_2}{2} \mathbf{w}_1 + \frac{\alpha_1 - \alpha_2}{2} \mathbf{w}_2.$$

So we have written the arbitrary vector $\mathbf{v} \in V$ as a linear combination of $\mathbf{w}_1, \mathbf{w}_2$; that is, $\mathbf{w}_1, \mathbf{w}_2$ is a spanning system.

Next we check the system is linearly independent. Suppose

$$\mathbf{0} = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2$$

for some scalars α_1, α_2. We must show $\alpha_1 = \alpha_2 = 0$. Using the formulas for the $\mathbf{w}_i$ in terms of the $\mathbf{v}_j$, the above equation becomes:

$$\mathbf{0} = \alpha_1 (\mathbf{v}_1 + \mathbf{v}_2) + \alpha_2 (\mathbf{v}_1 - \mathbf{v}_2) = (\alpha_1 + \alpha_2) \mathbf{v}_1 + (\alpha_1 - \alpha_2) \mathbf{v}_2.$$

Since $\mathbf{v}_1, \mathbf{v}_2$ forms a basis, the system is linearly independent. So it must be that $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 0$. Solving this system of equations yields $\alpha_1 = \alpha_2 = 0$, as needed. Thus $\mathbf{w}_1, \mathbf{w}_2$ is a linearly independent system. □

7. (a) This transformation is linear:

$$T\left(\alpha\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \beta\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = T\begin{pmatrix} \alpha x + \beta a \\ \alpha y + \beta b \\ \alpha z + \beta c \end{pmatrix} = \begin{pmatrix} \alpha x + \beta a + 3(\alpha y + \beta b) \\ -(\alpha z + \beta c) \end{pmatrix}$$

$$= \alpha\begin{pmatrix} x + 3y \\ -z \end{pmatrix} + \beta\begin{pmatrix} a + 3b \\ -c \end{pmatrix} = \alpha T\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \beta T\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

□

(b) This transformation is **not** linear. Consider $\mathbf{v} := (1,0,0)^T$. If T were linear, then we should have $T(2\mathbf{v}) = 2T(\mathbf{v})$. However,

$$T(2\mathbf{v}) = T\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2 + 4 = 6,$$

while

$$2T(\mathbf{v}) = 2T\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2(1+4) = 10.$$

Thus T is no linear. □

(c) This transformation is **not** linear. Consider $f(x) := x$. If T were linear, then we should have $T(2f) = 2T(f)$. However, $(2f)(x) = 2x$ and so

$$[T(2f)](x) = (2x)^2 = 4x^2,$$

while

$$[2T(f)](x) = 2[T(f)](x) = 2(x^2) = 2x^2.$$

These functions clearly are different, but to make this difference explicit note that the differ at $x = 1$ in particular. Thus T is not linear. □