Exercises:

1. Compute the following products:

   (a) $\begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

   (b) $\begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & 1 \\ 4 & 0 & 0 \\ 0 & -1 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 1 \\ 3 \end{pmatrix}$

2. For each of the following linear transformations T, find their matrix representations $[T]$.

   (a) $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is the linear transformation defined by $T(x,y)^T = (2x-y, y-3x, 4x)^T$.

   (b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that sends a vector $\mathbf{v} \in \mathbb{R}^2$ to its reflection over the line $y = x$.

   (c) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ projects every vector onto the x-y plane.

   (d) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ reflects every vector through the x-y plane.

   (e) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ rotates the x-y plane $\frac{\pi}{6}$ radians counterclockwise, but leaves the z-axis fixed.

3. Consider the following matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 1 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

   (a) Determine which of the following products are defined and give the dimension of the result: AB, BA, ABC, ABD, BC, BC^T, B^TC, DC, and D^TC^T.

   (b) Compute the following matrices: AB, $A(3B+C)$, B^TA, and A^TB.

4. Recall that for an angle θ, the linear transformation $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the plane by θ radians counterclockwise is given by

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

   Let ϕ be another angle. Use the fact $R_\theta R_\phi = R_{\theta+\phi}$ to derive the well-known trigonometric formulas for $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$.

5. Find linear transformation $A, B\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $AB = O$ but $BA \neq O$.

6. Let $A \in M_{n\times m}$ matrix. Define transformations $S, T\colon M_{m\times n} \to \mathbb{F}$ by $S(B) = \text{tr}(AB)$ and $T(B) = \text{tr}(BA)$. Prove that S and T are linear transformations, and that in fact $S = T$. Finally, use this to conclude that $\text{tr}(BC) = \text{tr}(CB)$ for any matrices $B \in M_{m\times n}$ and $C \in M_{n\times m}$.

7. The following are True/False. Prove the True statements and provide counterexamples for the False statements.

   (a) If $T\colon V \to W$ is a linear transformation and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent in V, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent in W.

   (b) If $T\colon V \to W$ is a linear transformation and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are such that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent in W, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent in V.

(c) Given $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^2$ such that $\mathbf{v}_1 \neq \mathbf{v}_2$, there exists a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.

Solutions:

1. (a) $\begin{pmatrix} 9 \\ -3 \end{pmatrix}$

   (b) $\begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix}$

   (c) Does not exist.

2. (a) $[T] = \begin{pmatrix} 2 & -1 \\ -3 & 1 \\ 4 & 0 \end{pmatrix}$

   (b) $[T] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   (c) $[T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

   (d) $[T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

   (e) $[T] = \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 0 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. (a) AB is 2×3, BA DNE, ABC DNE, ABD is 2×1, BC DNE, BC^T is 2×2, $B^T C$ is 3×3, DC DNE, $D^T C^T$ is 1×2.

   (b)

$$AB = \begin{pmatrix} 7 & 2 & -2 \\ 6 & 1 & 4 \end{pmatrix}$$

$$A(3B + C) = \begin{pmatrix} 18 & 6 & 5 \\ 19 & -2 & 20 \end{pmatrix}$$

$$A^T B = \begin{pmatrix} 10 & 3 & -4 \\ 5 & 1 & 2 \end{pmatrix}$$

4. We first compute the matrix product $R_\theta R_\phi$:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) & -\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\phi) \end{pmatrix}$$

Since rotating first by ϕ radians and then rotating by θ radians is the same as rotating by $\theta + \phi$ radians, the above matrix must equal

$$R_{\theta+\phi} = \begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix}.$$

In particular, the $(1, 1)$ and $(2, 1)$ entries must agree which says

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi).$$

5. Note that if $AB = O$, then for every vector $\mathbf{v} \in \mathbb{R}^2$, $AB(\mathbf{v}) = A(B\mathbf{v}) = \mathbf{0}$. So A must send $B(\mathbf{v})$ to $\mathbf{0}$. Thus we should look for a matrix A that sends lots of vectors to zero. On the other hand, we cannot take $A = O$ since this would imply $BA = BO = O$. A good compromise is letting A be the projection onto the x-axis. Then $A(x, y)^T = (x, 0)^T$. At this point it is useful to start thinking in terms of matrices. We have
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
since $A(1, 0)^T = (1, 0)$ and $A(0, 1)^T = (0, 0)^T$. Suppose
$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$
Then we have
$$AB = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix}.$$
Thus if choose $b_{11} = b_{12} = 0$ and $b_{21} = b_{22} = 1$ (for example). Then $AB = O$ while
$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq O.$$

6. Let $B, C \in M_{m\times n}$ and let β, γ be scalars. Then using the fact that tr is a linear transformation we have
$$S(\beta B + \gamma C) = \mathrm{tr}(A(\beta B + \gamma C)) = \mathrm{tr}(\beta AB + \gamma AC) = \beta \mathrm{tr}(AB) + \gamma \mathrm{tr}(AC) = \beta S(B) + \gamma S(C).$$
So S is linear. The proof for T is similar.

As linear transformations, S and T are completely determined by their outputs on a basis. Recall that if $E_{i,j}$ is the matrix with a 1 in its (i, j)th entry and zeros elsewhere, then $\{E_{i,j} : i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a basis for $M_{m\times n}$. So to see $S = T$, it suffices to show $S(E_{i,j}) = T(E_{i,j})$ for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Note that the (k, k) entry of $AE_{i,j}$ is
$$(A)_{k,1}(E_{i,j})_{1,k} + (A)_{k,2}(E_{i,j})_{2,k} + \cdots + (A)_{k,m}(E_{i,j})_{k,m}.$$
This is zero unless $k = j$, in which case the above reduces to $(A)_{j,i}(E_{i,j})_{i,j} = (A)_{j,i}$. Thus $S(E_{i,j}) = (A)_{j,i}$. On the other hand, the (k, k) entry of $E_{i,j}A$ is
$$(E_{i,j})_{k,1}(A)_{1,k} + (E_{i,j})_{k,2}(A)_{2,k} + \cdots + (E_{i,j})_{k,n}(A)_{n,k}.$$
This is zero unless $k = i$, in which case the above reduces to $(E_{i,j})_{i,j}(A)_{j,i} = (A)_{j,i}$. Thus $T(E_{i,j}) = (A)_{j,i}$, and so $S = T$.

This implies for any $B \in M_{m\times n}$ that
$$\mathrm{tr}(AB) = S(B) = T(B) = \mathrm{tr}(BA).$$
Since $A \in M_{n\times m}$ was arbitrary, the above equality holds for any $C \in M_{n\times m}$. □

7. (a) **False:** Let $V = \mathbb{R}^n$ and let $O : \mathbb{R}^n \to \mathbb{R}^n$ be the zero linear transformation. Then $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent since they form a basis for $\mathbb{R}^n$, but $O(\mathbf{e}_1) = \cdots = O(\mathbf{e}_n) = \mathbf{0}$ are linearly dependent:
$$1 \cdot O(\mathbf{e}_1) + \cdots + 1 \cdot O(\mathbf{e}_n) = \mathbf{0}.$$
□

(b) **True:** Suppose
$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0},$$

for some scalars $\alpha_1, \dots, \alpha_n$. Applying T to both sides and appealing to its linearity yields

$$\begin{aligned} T(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n) &= T(\mathbf{0}) \\ \alpha_1 T(\mathbf{v}_1) + \cdots + \alpha_n T(\mathbf{v}_n) &= \mathbf{0}. \end{aligned}$$

Since $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$ is a linearly independent system, we must have $\alpha_1 = \cdots = \alpha_n$. Thus $\mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent. □

(c) **False:** Consider

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad \mathbf{w}_1 = \mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Observe that $\mathbf{v}_2 = 2\mathbf{v}_1$ while $\mathbf{w}_2 \neq 2\mathbf{w}_1$. Suppose, towards a contradiction that there is a linear transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. It would then follow that

$$\mathbf{w}_2 = T(\mathbf{v}_2) = T(2\mathbf{v}_1) = 2T(\mathbf{v}_1) = 2\mathbf{w}_1,$$

or $\mathbf{w}_2 = 2\mathbf{w}_1$, a contradiction. So no such linear transformation exists. □