Exercises:

1. Let $T\colon V \to W$ be an isomorphism, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$.

   (a) Prove that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a generating system in V, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is a generating system in W.

   (b) Prove that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a linearly independent system in V, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is a linearly independent system in W.

   (c) Prove that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for V, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is a basis for in W.

2. Find all right inverses of the matrix $A = (1, 1) \in M_{1\times 2}$. Use this to prove that A is **not** left invertible.

3. Suppose $A\colon V \to W$ and $B\colon U \to V$ are linear transformations such that $A \circ B$ is invertible.

   (a) Prove that A is right invertible and B is left invertible.

   (b) Find an example of such an A and B so that A is **not** left invertible and B is **not** right invertible.

4. Let $T\colon V \to W$ be a linear transformation.

   (a) Show that
   $$\text{Null}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$$
   is a subspace of V.

   (b) Show that
   $$\text{Ran}(T) = \{\mathbf{w} \in W : \text{there exists } \mathbf{v} \in V \text{ such that } T(\mathbf{v}) = \mathbf{w}\}$$
   is a subspace of W.

   (c) Prove that T is an isomorphism if and only if $\text{Null}(T) = \{\mathbf{0}_V\}$ (the trivial subspace) and $\text{Ran}(T) = W$.

5. Let $X, Y \subset V$ be subspaces of V.

   (a) Show that $X \cap Y$ is a subspace of V.

   (b) Show that $X \cup Y$ is a subspace of V if and only if either $X \subset Y$ or $Y \subset X$.

6. Recall that $\mathcal{L}(V, W)$ denotes the space of linear transformations from V to W. Consider the following subset
   $$\mathcal{IL}(V, W) := \{T \in \mathcal{L}(V, W) : T \text{ is invertible}\}.$$
   Show that $\mathcal{IL}(V, W)$ is a subspace if and only if $O \in \mathcal{IL}(V, W)$ where $O\colon V \to W$ is the trivial linear transformation defined by $O(\mathbf{v}) = \mathbf{0}_W$ for all $\mathbf{v} \in V$.

   [**Hint:** for the “if” direction think about what it means for V and W if O is invertible.]

Solutions:

1. (a) Let $\mathbf{w} \in W$ be an arbitrary vector. We must show there exists scalars $\alpha_1, \ldots, \alpha_n$ such that
   $$\mathbf{w} = \sum_{j=1}^{n} \alpha_j T(\mathbf{v}_j).$$
   Note that since T is invertible, its inverse $T^{-1}\colon W \to V$ exists. Then $T^{-1}(\mathbf{w}) \in V$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are generating, there exists scalars $\beta_1, \ldots, \beta_n$ such that
   $$T^{-1}(\mathbf{w}) = \sum_{j=1}^{n} \beta_j \mathbf{v}_j.$$

Applying T to each side yields

$$T(T^{-1}(\mathbf{w})) = T\left(\sum_{j=1}^{n} \beta_j \mathbf{v}_j\right)$$
$$\mathbf{w} = \sum_{j=1}^{n} \beta_j T(\mathbf{v}_j),$$

where on the left we have used the definition of the inverse, and on the right we have used the linearity of T. Thus if we choose $\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$, then we have the desired equality. Since $\mathbf{w} \in W$ was arbitrary, this shows that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is a generating system in W. □

(b) Suppose there are scalars $\alpha_1, \ldots, \alpha_n$ such that

$$\sum_{j=1}^{n} \alpha_j T(\mathbf{v}_j) = \mathbf{0}.$$

We must show $\alpha_1 = \cdots = \alpha_n = 0$. Using the linearity of T, this is equivalent to

$$T\left(\sum_{j=1}^{n} \alpha_j \mathbf{v}_j\right) = \mathbf{0}.$$

Applying T^{-1} to each side then yields

$$\sum_{j=1}^{n} \alpha_j \mathbf{v}_j = \mathbf{0},$$

since $T^{-1}(\mathbf{0}) = \mathbf{0}$. Now, since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a linearly independent system, the above equation implies that we must have $\alpha_1 = \cdots = \alpha_n = 0$, as needed. □

(c) Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis. Then the system is generating and linearly independent. Therefore parts (a) and (b) imply the system $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is generating and linearly independent (respectively), and consequently a basis. □

2. A right inverse of A is a matrix $B \in M_{2\times 1}$ such that $AB = I_1 = (1)$. Label the entries of $B = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

$$AB = (a + b).$$

Thus we require $a + b = 1$ or $b = 1 - a$. That is, the right inverses of A look like $\begin{pmatrix} a \\ 1-a \end{pmatrix}$ for any scalar a.

Suppose, towards a contradiction, that A is left invertible. Then A is both left and right invertible and hence invertible. However, we showed in lecture that this means the left and right inverses of A are the same, and moreover are **unique**. This is a contradiction because choosing $a = 0$ and $a = 1$ yield two distinct right inverses for A. Thus it must be that A is **not** left invertible. □

3. (a) Consider the following equations which hold by virtue of the invertibility of $A \circ B$:

$$(A \circ B) \circ (A \circ B)^{-1} = I_W$$
$$(A \circ B)^{-1} \circ (A \circ B) = I_U.$$

Since the composition of linear transformations is associative, we can move the parentheses above around to get:

$$A \circ \left(B \circ (A \circ B)^{-1}\right) = I_W$$
$$\left((A \circ B)^{-1} \circ A\right) \circ B = I_U.$$

The first equation is precisely saying that $B \circ (A \circ B)^{-1}$ is the right inverse of A, while the second equation says $(A \circ B)^{-1} \circ A$ is the left inverse of B. Thus A is right invertible and B is left invertible. □

(b) Let $A = (1, 1) \in M_{1\times 2}$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $AB = (1) = I_1$, which is invertible, but as was shown in the previous exercise A is **not** left invertible. Consequently, B cannot be right invertible because if it was, then its right inverse would necessarily be A, but this would contradict A not being left invertible. □

4. (a) We know $T(\mathbf{0}_V) = \mathbf{0}_W$ since T is a linear transformation. Thus $\mathbf{0}_V \in \text{Null}(T)$. Next, let $\mathbf{v}, \mathbf{w} \in \text{Null}(T)$. Then since T is linear we have

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W,$$

so that $\mathbf{v} + \mathbf{w} \in \text{Null}(T)$. Finally, for $\mathbf{v} \in \text{Null}(T)$ and a scalar α we have

$$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}) = \alpha\mathbf{0}_W = \mathbf{0}_W,$$

where we have again used the linearity of T along with Exercise 3 on Homework 1. Thus $\alpha\mathbf{v} \in \text{Null}(T)$, and so $\text{Null}(T)$ is a subspace.

(b) Since $T(\mathbf{0}_V) = \mathbf{0}_W$, we know $\mathbf{0}_W \in \text{Ran}(T)$. Let $\mathbf{w}_1, \mathbf{w}_2 \in \text{Ran}(T)$. Then there exists $\mathbf{v}_1, \mathbf{v}_2 \in V$ so that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. So using the linearity of T we have

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2,$$

which means $\mathbf{w}_1 + \mathbf{w}_2 \in \text{Ran}(T)$. Finally, let $\mathbf{w} \in \text{Ran}(T)$ and let α be a scalar. There exists $\mathbf{v} \in V$ so that $T(\mathbf{v}) = \mathbf{w}$, and therefore $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}) = \alpha\mathbf{w}$. Thus $\alpha\mathbf{w} \in \text{Ran}(T)$, and $\text{Ran}(T)$ is a subspace.

(c) ($\Longrightarrow$): Assume that T is an isomorphism. Suppose $\mathbf{v} \in \text{Null}(T)$, so that $T(\mathbf{v}) = \mathbf{0}_W$. Since T is invertible, we can apply T^{-1} to each side, which yields $\mathbf{v} = T^{-1}(\mathbf{0}_W)$. But since T^{-1} is linear, we must have $T^{-1}(\mathbf{0}_W) = \mathbf{0}_V$, and so $\mathbf{v} = \mathbf{0}_V$. Thus $\text{Null}(T) \subset \{\mathbf{0}_V\}$. On the other hand, $\mathbf{0}_V$ is always in the null space of T so we in fact have $\text{Null}(T) = \{\mathbf{0}_V\}$.

Next let $\mathbf{w} \in W$ be arbitrary. Define $\mathbf{v} := T^{-1}(\mathbf{w}) \in V$. Then $T(\mathbf{v}) = T(T^{-1}(\mathbf{w})) = \mathbf{w}$, which means $\mathbf{w} in \text{Ran}(T)$. Since $\mathbf{w}$ was arbitrary, this implies $W \subset \text{Ran}(T)$. The other inclusion follows by definition of the range, so we therefore have $\text{Ran}(T) = W$.

($\Longleftarrow$): Assume $\text{Null}(T) = \{\mathbf{0}_V\}$ and $\text{Ran}(T) = W$. We will define a linear transformation $S \colon W \to V$ and show that it is the inverse of T. We first make an observation. Let $\mathbf{w} \in W = \text{Ran}(T)$, then there exists at least one $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. We claim that this $\mathbf{v}$ is in fact unique. Indeed, suppose there is another $\mathbf{v}' \in V$ such that $T(\mathbf{v}') = \mathbf{w}$. Then

$$T(\mathbf{v} - \mathbf{v}') = T(\mathbf{v}) - T(\mathbf{v}') = \mathbf{w} - \mathbf{w} = \mathbf{O}_W.$$

This means $\mathbf{v} - \mathbf{v}' \in \text{Null}(T)$, but since the null space is the trivial subspace we must have $\mathbf{v} - \mathbf{v}' = \mathbf{0}_V$, or equivalently $\mathbf{v} = \mathbf{v}'$. Thus $\mathbf{v}$ is unique.

We therefore define a transformation $S \colon W \to V$ by letting $S(\mathbf{w})$ equal the unique $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. We must check that S is linear and that it is the inverse of T. We check the latter condition first. Let $\mathbf{v} \in V$ be arbitrary and call $\mathbf{w} := T(\mathbf{v})$. By definition of S, we have $S(\mathbf{w}) = \mathbf{v}$ and so

$$S \circ T(\mathbf{v}) = S(T(\mathbf{v})) = S(\mathbf{w}) = \mathbf{v}.$$

Since $\mathbf{v} \in V$ was arbitrary, this implies $S \circ T = I_V$. Next let $\mathbf{w} \in W$ be arbitrary and call $\mathbf{v} := S(\mathbf{w})$. Again, by definition of S we must have $T(\mathbf{v}) = \mathbf{w}$. Thus

$$T \circ S(\mathbf{w}) = T(S(\mathbf{w})) = T(\mathbf{v}) = \mathbf{w}.$$

Since $\mathbf{w} \in W$ was arbitrary, this implies $T \circ S = I_W$. Finally, we verify that S is linear. Let $\mathbf{w}_1, \mathbf{w}_2 \in W$ and let α, β be scalars. Call $v := S(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2)$, $\mathbf{v}_1 := S(\mathbf{w}_1)$, and $\mathbf{v}_2 := S(\mathbf{w}_2)$. We must show $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$. Observe that by the linearity of T and the definition of S we have

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) = \alpha\mathbf{w}_1 + \beta\mathbf{w}_2.$$

Since $\mathbf{v}$ is defined to be the unique vector satisfying $T(\mathbf{v}) = \alpha\mathbf{w}_1 + \beta\mathbf{w}_2$, we must have $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ as desired. □

5. (a) Since X and Y are both subspaces, we have $\mathbf{0}_V \in X$ and $\mathbf{0}_V \in Y$. Thus $\mathbf{0}_V \in X \cap Y$. Let $\mathbf{v}, \mathbf{w} \in X \cap Y$. Then $\mathbf{v}, \mathbf{w} \in X$ and since X is subspace we have $\mathbf{v} + \mathbf{w} \in X$. Similarly, $\mathbf{v}, \mathbf{w} \in Y$ so that $\mathbf{v} + \mathbf{w} \in Y$. Thus $\mathbf{v} + \mathbf{w} \in X \cap Y$. Finally, let $\mathbf{v} \in X \cap Y$ and let α be a scalar. Then $\mathbf{v} \in X$ and so $\alpha\mathbf{v} \in X$. Similarly, $\mathbf{v} \in Y$ and so $\alpha\mathbf{v} \in Y$. Hence $\alpha\mathbf{v} \in X \cap Y$, which means $X \cap Y$ is a subspace. □

   (b) ($\Longrightarrow$): We will use proof by contrapositive to show this. That is, we will assume that neither $X \subset Y$ nor $Y \subset X$ and then show that $X \cup Y$ is **not** a subspace. In particular, we will show that $X \cup Y$ is not closed under addition. Now, since X is not contained in Y there exists some $\mathbf{v} \in X$ such that $\mathbf{v} \notin Y$. Similarly, since Y is not contained in X there is some $\mathbf{w} \in Y$ such that $\mathbf{w} \in\notin X$. Now, $\mathbf{v}, \mathbf{w} \in X \cup Y$ but we claim $\mathbf{v} + \mathbf{w} \notin X \cup Y$. Indeed, if we had $\mathbf{v} + \mathbf{w} \in X \cup Y$ then it would follow that either $\mathbf{v} + \mathbf{w} \in X$ or $\mathbf{v} + \mathbf{w} \in Y$ (or both). If $\mathbf{v} + \mathbf{w} \in X$ then there is some $\mathbf{x} \in X$ such that $\mathbf{v} + \mathbf{w} = \mathbf{x}$. But then $\mathbf{w} = \mathbf{x} - \mathbf{v}$ and the difference on the right is in X since both $\mathbf{x}$ and $\mathbf{v}$ are. This contradicts $\mathbf{w} \notin X$. So we cannot have $\mathbf{v} + \mathbf{w} \in X$. On the other hand, if we have $\mathbf{v} + \mathbf{w} \in Y$ then there is some $\mathbf{y} \in Y$ such that $\mathbf{v} + \mathbf{w} = \mathbf{y}$. But then $\mathbf{v} = \mathbf{y} - \mathbf{w}$, which is in Y since both $\mathbf{y}$ and $\mathbf{w}$ are. This contradicts $\mathbf{v} \notin Y$. Since we cannot have either $\mathbf{v} + \mathbf{w} \in X$ or $\mathbf{v} + \mathbf{w} \in Y$, we must have $\mathbf{v} + \mathbf{w} \notin X \cup Y$. So $X \cup Y$ is not a subspace.

   ($\Longleftarrow$): If $X \subset Y$ or $Y \subset X$, then $X \cup Y$ is either Y or X, respectively. In either case, $X \cup Y$ is a subspace. □

6. ($\Longrightarrow$): If $\mathcal{IL}(V, W)$ is a subspace, then it necessarily contains the zero vector, which in $\mathcal{L}(V, W)$ is the trivial linear transformation O.

   ($\Longleftarrow$): Assume $O \in \mathcal{IL}(V, W)$. Let O^{-1} be the inverse of O. Observe that for any $\mathbf{v} \in V$ we have

$$\mathbf{v} = O^{-1} \circ O(\mathbf{v}) = O^{-1}(O(\mathbf{v})) = O^{-1}(\mathbf{0}_W) = \mathbf{0}_V,$$

   since O^{-1} is a linear transformation. Thus $\mathbf{v} \in \mathbf{0}_V$, and since this was an arbitrary vector we therefore have $V = \{\mathbf{0}_V\}$ is a vector space with exactly one element. Similarly, for any $\mathbf{w} \in W$ we have

$$\mathbf{w} = O \circ O^{-1}(\mathbf{w}) = O(O^{-1}(\mathbf{w})) = \mathbf{0}_W.$$

   So $\mathbf{w} = \mathbf{0}_W$ and by the same reasoning as above we get that $W = \{\mathbf{0}_W\}$. Now, consider any $T \in \mathcal{L}(V, W)$. Then $T = O$ since they agree on all vectors of V: $T(\mathbf{0}_V) = \mathbf{0}_W = O(\mathbf{0}_V)$. Consequently $\mathcal{L}(V, W) = \{O\}$ is a vector space with exactly one element. It follows that $\mathcal{IL}(V, W) = \{O\}$, and so we see that it is a subspace; namely, the trivial subspace. □