Exercises:

1. The reduced row echelon form of a matrix A is
$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$
If the first, second, and fourth columns of A are
$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$
respectively, find the original matrix A.

2. Prove whether or not the system of polynomials
$$p_1(x) = x^3 + 2x, \qquad p_2(x) = x^2 + x + 1, \qquad p_3(x) = x^3 + 5,$$
generates $\mathbb{P}_3$, the vector space of polynomials with degree at most three.

3. Suppose $A\mathbf{x} = \mathbf{0}$ has a unique solution. Prove that A is left invertible.

4. Compute the inverses of the following matrices (if they exist). Show your work.

   (a) $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 4 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 & 3 \\ 3 & -1 & 0 \\ 4 & -1 & 3 \end{pmatrix}$

5. Let V be a finite-dimensional vector space with $\dim(V) = n$. Show that a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ is linearly independent if and only if it is spanning in V.

6. Define a linear transformation $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ by $T(p(x)) = p(x) - p'(x)$. Determine whether or not T is invertible. If it is, write down a formula for its inverse. If it is not, provide a reason.

7. Find a 2×3 linear system of equations whose general solution is
$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Solutions:

1. Denote the reduced row echelon form of A by B. Recall that to solve $A\mathbf{x} = \mathbf{0}$, we conduct row operations on $(A \mid \mathbf{0})$ until we obtain $(B \mid \mathbf{0})$. So we see that solutions of $A\mathbf{x} = \mathbf{0}$ are of the following form:
$$\mathbf{x} = \begin{pmatrix} -2x_3 + 2x_5 \\ 5x_3 + 3x_5 \\ x_3 \\ -6x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 3 \\ 0 \\ -6 \\ 1 \end{pmatrix}, \qquad x_3, x_5 \in \mathbf{F}.$$

Now, denote the columns of A by $\mathbf{v}_1, \ldots, \mathbf{v}_5$. Choosing $x_3 = 1$ and $x_5 = 0$ above, implies

$$-2\mathbf{v}_1 + 5\mathbf{v}_2 + 1\mathbf{v}_3 + 0\mathbf{v}_4 + 0\mathbf{v}_5 = \mathbf{0}.$$

We already know $\mathbf{v}_1$ and $\mathbf{v}_2$, so suppose $\mathbf{v}_3 = (a, b, c)^T$. Then the above implies

$$\begin{aligned} -2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + 5\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -2+a \\ -3+b \\ -1+c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus $a = 2$, $b = 3$, and $c = 1$ so that

$$\mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Next, choosing $x_3 = 0$ and x_5 above yields

$$2\mathbf{v}_1 + 3\mathbf{v}_2 + 0\mathbf{v}_3 - 6\mathbf{v}_4 + 1\mathbf{v}_5 = \mathbf{0}.$$

Suppose $\mathbf{v}_5 = (x, y, z)^T$. Then the above equation becomes

$$\begin{aligned} 2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - 6\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -4+x \\ 7+y \\ 9+z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus $x = 4$, $y = -7$, and $z = -9$ so that

$$\mathbf{v}_5 = \begin{pmatrix} 4 \\ -7 \\ -9 \end{pmatrix}.$$

2. Recall that $x^3, x^2, x, 1$ forms a basis for $\mathbb{P}_3$. Hence $\dim(\mathbb{P}_3) = 4$. By a corollary from lecture, we therefore know that any spanning system for $\mathbb{P}_3$ must contain at least 4 vectors. Consequently, $p_1(x), p_2(x), p_3(x)$ is **not** spanning for $\mathbb{P}_3$. $\square$

3. Suppose $A \in M_{m\times n}$. Recall that by an observation from lecture that a solution is unique iff the row echelon form has a pivot in every column. Since the pivot rows and columns are the same in the *reduced* row echelon form, we see that that RREF of A (call it B) has a pivot in every column. It follows that there must be more rows than columns; that is, $m \geq n$. Moreover, B must look like

$$B = \begin{pmatrix} I_n \\ 0 \cdots 0 \\ \vdots \\ 0 \cdots 0 \end{pmatrix}.$$

Also recall that $B = EA$ for $E \in M_{m\times m}$ a product of elementary matrices. Let $E_0 \in M_{n\times m}$ be matrix obtained from E after deleting the last $m - n$ rows of E. We claim that $E_0 A = I_n$, so that A is left invertible. Indeed, $E_0 A$ has entries

$$(E_0 A)_{i,j} = \sum_{k=1}^{m} (E_0)_{i,k}(A)_{k,j} = \sum_{k=1}^{m} (E)_{i,k}(A)_{k,j} = (EA)_{i,j} = (B)_{i,j},$$

for $i, j = 1, \ldots, n$. Since these entries of B are precisely I_n, we have $E_0 A = I_n$ as claimed. $\square$

4. (a) $\frac{1}{2}\begin{pmatrix} 19 & -5 & -1 \\ -6 & 2 & 0 \\ -5 & 1 & 1 \end{pmatrix}$

   (b) $\frac{1}{12}\begin{pmatrix} 6 & 5 & 1 \\ -6 & 3 & 3 \\ 0 & -2 & 2 \end{pmatrix}$

   (c) The inverse does not exist.

5. ($\Longrightarrow$): Assume $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a linearly independent system. If it is not spanning, then by Exercise 5 on Homework 2 we can find a vector $\mathbf{v}_{n+1} \in V$ so that $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}$ is linearly independent. But since $\dim(V) = n$, this contradicts a result from lecture that says a linearly independent set in V can have no more than n vectors. Thus it must be the case that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is spanning.

   ($\Longleftarrow$): Assume $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a spanning system in V. Recall from lecture that any finite spanning set can be reduced to a basis. So if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is not linearly independent then we can reduce it to a basis with strictly fewer than n vectors. But this contradicts $\dim(V) = n$. Thus it must be the case that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are already linearly independent. $\square$

6. We know how to compute the inverse of a matrix (if it exists), so we must somehow change this problem into one about matrices. Recall that every linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$ is equivalent to an $m \times n$ matrix, so we can bring matrices into the picture by replacing $\mathbb{P}_2$ with $\mathbb{F}^n$ for some n. Recall that $\mathbb{P}_2 \cong \mathbb{F}^3$, and in particular we have an isomorphism $S\colon \mathbb{F}^3 \to \mathbb{P}_2$ defined by

$$S(\mathbf{e}_1) = 1, \qquad S(\mathbf{e}_2) = x, \qquad S(\mathbf{e}_3) = x^2.$$

Then $S^{-1} \circ T \circ S\colon \mathbb{F}^3 \to \mathbb{F}^3$ and so it is equivalent to the matrix whose columns we determine by feeding in the standard basis for $\mathbb{F}^3$. We have

$$\begin{aligned} S^{-1} \circ T \circ S(\mathbf{e}_1) &= S^{-1}(T(1)) = S^{-1}(1) = \mathbf{e}_1 \\ S^{-1} \circ T \circ S(\mathbf{e}_2) &= S^{-1}(T(x)) = S^{-1}(x-1) = \mathbf{e}_2 - \mathbf{e}_1 \\ S^{-1} \circ T \circ S(\mathbf{e}_3) &= S^{-1}(T(x^2)) = S^{-1}(x^2 - 2x) = \mathbf{e}_3 - 2\mathbf{e}_2. \end{aligned}$$

Thus

$$S^{-1} \circ T \circ S = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Row operations then yield

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

So $S^{-1} \circ T \circ S$ is invertible with inverse

$$B := \left(S^{-1} \circ T \circ S\right)^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We claim $R := S \circ B \circ S^{-1}$ is the inverse of T. Note that $R\colon \mathbb{P}_2 \to \mathbb{P}_2$, and it is defined on the standard basis of $\mathbb{P}_2$ as follows:

$$\begin{aligned} R(1) &= S(B(\mathbf{e}_1)) = S(\mathbf{e}_1) = 1 \\ R(x) &= S(B(\mathbf{e}_2)) = S(\mathbf{e}_1 + \mathbf{e}_2) = 1 + x \\ R(x^2) &= S(B(\mathbf{e}_3)) = S(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3) = 2 + 2x + x^2. \end{aligned}$$

It suffices to check $R \circ T = I$ and $T \circ R = I$ on the standard basis. We have

$$\begin{aligned} R \circ T(1) &= R(1) = 1 \\ R \circ T(x) &= R(x-1) = R(x) - R(1) = 1 + x - 1 = x \\ R \circ T(x^2) &= R(x^2 - 2x) = R(x^2) - 2R(x) = 2 + 2x + x^2 - 2(1+x) = x^2, \end{aligned}$$

so that $R \circ T = I$. Similarly, we have

$$\begin{aligned} T \circ R(1) &= T(1) = 1 \\ T \circ R(x) &= T(1+x) = 1 + x - 1 = x \\ T \circ R(x^2) &= T(2 + 2x + x^2) = 2 + 2x + x^2 - (2 + 2x) = x^2, \end{aligned}$$

so that $T \circ R = I$. Thus $R = T^{-1}$. □

7. (Note that there are many correct answers to this problem.) We must find a matrix $A \in M_{2\times 3}(\mathbb{R})$ and a vector $\mathbf{b} \in \mathbb{R}^2$ such that all solutions of the system $A\mathbf{x} = \mathbf{b}$ have the given form. Recall from lecture that $(1,1,0)^T$ is a particular solution of the system, while $(1,2,1)^T$ is a solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. That is, $(1,2,1)^T \in \mathrm{Ker}(A)$. Moreover, $(1,2,1)^T$ should for a basis for $\mathrm{Ker}(A)$. Since the dimension of the kernel is determined by the number of free variables, this means we should have exactly one free variable. Without loss of generality, we can assume it is x_3 and solutions of $A\mathbf{x} = 0$ are of the form

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_3 \\ 2x_3 \\ x_3 \end{pmatrix}, \qquad x_3 \in \mathbb{R}.$$

We can see that this corresponds to the following RREF for A:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Now, we may as well assume A is equal to its own RREF, since this will not change the solutions to the homogeneous system. So we take

$$A := \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix},$$

and it remains to determine $\mathbf{b}$. Since we know $(1,1,0)^T$ is a particular solution, this gives $\mathbf{b}$:

$$\mathbf{b} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Conveniently, $(A \mid \mathbf{b})$ is *already* in RREF, and so we can easily check that our answer yields the prescribed general solution.