Exercises:

1. For the following matrix, compute its rank and find bases for each of its four fundamental subspaces:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let $S\colon U \to V$ and $T\colon V \to W$ be a linear transformations between finite-dimensional vector spaces.
   (a) Prove that if $V_0 \subset V$ is a subspace, then
   $$T(V_0) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V_0\}$$
   is a subspace.
   (b) Prove that $\dim(T(V_0)) \leq \min\{\text{rank}(T), \dim(V_0)\}$.
   (c) Prove that $\text{rank}(T \circ S) \leq \min\{\text{rank}(T), \text{rank}(S)\}$.

3. Let V be a finite dimensional vector space and let $X, Y \subset V$ be subspaces. The goal of this exercise is to proof the following formula
$$\dim(X + Y) = \dim(X) + \dim(Y) - \dim(X \cap Y).$$
Here, $X + Y := \{\mathbf{v} = \mathbf{x} + \mathbf{y} : \mathbf{x} \in X,\ \mathbf{y} \in Y\}$.
   (a) Prove that $X + Y$ is a subspace of V.
   (b) The **direct sum** of X and Y is the following set
   $$X \oplus Y := \{(x, y) : x \in X,\ y \in Y\}.$$
   This can be made into a vector space with operations of addition
   $$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
   and scalar multiplication
   $$\alpha(x, y) = (\alpha x, \alpha y).$$
   You do not need to prove that $X \oplus Y$ is a vector space, but do prove that $\dim(X \oplus Y) = \dim(X) + \dim(Y)$.
   (c) Consider the transformation $T\colon X \oplus Y \to V$ defined by $T(x, y) = x - y$. Prove that T is linear.
   (d) Show that $\text{Ran}(T) = X + Y$.
   (e) Show that $\text{Ker}(T) \cong X \cap Y$.
   (f) Use the rank-nullity theorem on T to prove the claimed formula.

4. Let $A \in M_{m \times n}$. Prove that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{F}^m$ if and only if $A^T\mathbf{x} = \mathbf{0}$ has a unique solution.

5. Complete the following set of vectors to a basis for $\mathbb{R}^5$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 5 \\ 5 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} -1 \\ -4 \\ 4 \\ 7 \\ -8 \end{pmatrix}$$

Solutions:

1. The RREF of A is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & 1/2 \\ 0 & 0 & 1 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivots, we have $\text{rank}(A) = 3$. The pivots occur in the first three columns of B, so the corresponding columns of A form a basis for $\text{Ran}(A)$:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -3 \\ 0 \end{pmatrix}.$$

The pivots also occur in the first three rows of B, so those rows form a basis for $\text{Ran}(A^T)$:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1/4 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1/6 \\ 0 \end{pmatrix}.$$

To find a basis for $\text{Ker}(A)$, we note that the augmented matrix $(B \mid \mathbf{0})$ implies the solution of $A\mathbf{x} = \mathbf{0}$ are of the form:

$$\mathbf{x} = \begin{pmatrix} 0 \\ -\frac{1}{4}x_4 - \frac{1}{2}x_5 \\ -\frac{1}{6}x_4 \\ x_4 \\ x_5 \end{pmatrix} = x_4 \begin{pmatrix} 0 \\ -1/4 \\ -1//6 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad x_4, x_5 \in \mathbb{F}.$$

Thus $(0, -\frac{1}{4}, -\frac{1}{6}, 1, 0)^T, (0, -\frac{1}{2}, 0, 0, 1)^T$ form a basis for $\text{Ker}(A)$.
To find a basis for $\text{Ker}(A^T)$, we first compute the RREF of A^T:

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So solutions to $A^T\mathbf{x} = \mathbf{0}$ are of the form

$$\mathbf{x} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore $(1, -1, 1, 0)^T$ is a basis for $\text{Ker}(A^T)$.

2. (a) Suppose V_0 is a subpace. Then $\mathbf{0}_V \in V_0$ and therefore

$$\mathbf{0}_W = T(\mathbf{0}_V) \in T(V_0).$$

Next, let $\mathbf{w}_1, \mathbf{w}_2 \in T(V_0)$. Then there exists $\mathbf{v}_1, \mathbf{v}_2 \in V_0$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. But then using the linearity of T we have

$$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2).$$

Since V_0 is a subspace, $\mathbf{v}_1 + \mathbf{v}_2 \in V_0$ and so $\mathbf{w}_1 + \mathbf{w}_2 \in T(V_0)$. Finally, let α be a scalar and let $\mathbf{w}_1, \mathbf{v}_1$ be as above. Again using the linearity of T and the fact that V_0 is a subspace, we have

$$\alpha \mathbf{w}_1 = \alpha T(\mathbf{v}_1) = T(\alpha \mathbf{v}_1) \in T(V_0).$$

Thus $T(V_0)$ is a subspace. □

(b) We will show $\dim(T(V_0)) \leq \operatorname{rank}(T)$ and $\dim(T(V_0)) \leq \dim(V_0)$, which implies the desired inequality. First, note that $V_0 \subset V$ implies $T(V_0) \subset T(V)$. Thus $T(V_0)$ is a subspace (by the previous part) of $T(V)$. But $T(V) = \operatorname{Ran}(T)$ and so

$$\dim(T(V_0)) \leq \dim(T(V)) = \dim(\operatorname{Ran}(T)) = \operatorname{rank}(T).$$

Next, note that V_0 is finite-dimensional by virtue of being a subspace of the finite-dimensional vector space V. So let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for V_0. Then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_r)$ is spanning for $T(V_0)$. Indeed, given any $\mathbf{w} \in T(V_0)$ we have $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V_0$. But then there are scalars $\alpha_1, \ldots, \alpha_r$ such that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r$. Applying T to each side and using linearity we see that

$$\mathbf{w} = T(\mathbf{v}) = T(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r) = \alpha_1 T(\mathbf{v}_1) + \cdots + \alpha_r T(\mathbf{v}_r).$$

Thus $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_r)$ are spanning for $T(V_0)$ as claimed. Consequently, we know that this system can be reduced to a basis for $T(V_0)$. But then this system will have at most r vectors which implies

$$\dim(T(V_0)) \leq r = \dim(V_0),$$

as claimed. □

(c) Denote $V_0 := S(U) = \operatorname{Ran}(S)$. Then $T(V_0) = T(S(U)) = \operatorname{Ran}(T \circ S)$. Thus

$$\operatorname{rank}(T \circ S) = \dim(T(V_0))$$
$$\operatorname{rank}(S) = \dim(V_0).$$

Therefore the desired inequality follows immediately from the previous part. □

3. (a) Since both X and Y are subspaces, $\mathbf{0}_V \in X$ and $\mathbf{0}_V \in Y$. Therefore

$$\mathbf{0}_V = \mathbf{0}_V + \mathbf{0}_V \in X + Y.$$

Next, let $\mathbf{v}_1, \mathbf{v}_2 \in X + Y$. Then there are $\mathbf{x}_1, \mathbf{x}_2 \in X$ and $\mathbf{y}_1, \mathbf{y}_2 \in Y$ such that

$$\mathbf{v}_1 = \mathbf{x}_1 + \mathbf{y}_1 \qquad \text{and} \qquad \mathbf{v}_2 = \mathbf{x}_2 + \mathbf{y}_2.$$

But then since X and Y are closed under addition we have

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{x}_1 + \mathbf{y}_1 + \mathbf{x}_2 + \mathbf{y}_2 = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) \in X + Y.$$

Finally, let α be a scalar and let $\mathbf{v}_1, \mathbf{x}_1, \mathbf{y}_1$ be as above. Since X and Y are closed under scalar multiplication, we have (by the distributive law) that

$$\alpha \mathbf{v}_1 = \alpha(\mathbf{x}_1 + \mathbf{y}_1) = \alpha \mathbf{x}_1 + \alpha \mathbf{y}_1 \in X + Y.$$

Thus $X + Y$ is a subspace. □

(b) Since X and Y are subspaces of the finite-dimensional V, they are both finite-dimensional. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and $\mathbf{y}_1, \ldots, \mathbf{y}_m$ be bases for X and Y, respectively. We claim that

$$(\mathbf{x}_1, \mathbf{0}), \ldots, (\mathbf{x}_n, \mathbf{0}), (\mathbf{0}, \mathbf{y}_1), \ldots, (\mathbf{0}, \mathbf{y}_m)$$

is a basis for $X \oplus Y$. If this is true, then we have $\dim(X \oplus Y) = n + m = \dim(X) + \dim(Y)$, as desired. We first check linear independence: suppose there are scalars $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ such that

$$\alpha_1(\mathbf{x}_1, \mathbf{0}) + \cdots + \alpha_n(\mathbf{x}_n, \mathbf{0}) + \beta_1(\mathbf{0}, \mathbf{y}_1) + \cdots + \beta_m(\mathbf{0}, \mathbf{y}_m) = \mathbf{0}_{X \oplus Y}.$$

Note that $\mathbf{0}_{X\oplus Y} = (\mathbf{0},\mathbf{0})$. By definition of the addition and scalar multiplication operations, this is equivalent to

$$(\alpha_1\mathbf{x}_1 + \cdots \alpha_n\mathbf{x}_n, \beta_1\mathbf{y}_1 + \cdots + \beta_m\mathbf{y}_m) = (\mathbf{0},\mathbf{0}),$$

which is in turn equivalent to the pair of equations

$$\begin{aligned}\alpha_1\mathbf{x}_1 + \cdots \alpha_n\mathbf{x}_n &= \mathbf{0}\\ \beta_1\mathbf{y}_1 + \cdots + \beta_m\mathbf{y}_m &= \mathbf{0}.\end{aligned}$$

As $\mathbf{x}_1,\ldots,\mathbf{x}_n$ and $\mathbf{y}_1,\ldots,\mathbf{y}_m$ are both bases, we must have $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_m = 0$. Hence the original vectors are linearly independent. To see that they are spanning, let $(\mathbf{x},\mathbf{y}) \in X \oplus Y$. Then there are scalars $\alpha_1,\ldots,\alpha_n$ and scalars $\beta_1,\ldots,\beta_m$ such that

$$\begin{aligned}\alpha_1\mathbf{x}_1 + \cdots \alpha_n\mathbf{x}_n &= \mathbf{x}\\ \beta_1\mathbf{y}_1 + \cdots + \beta_m\mathbf{y}_m &= \mathbf{y}.\end{aligned}$$

But then by the same computation as above, we have

$$\alpha_1(\mathbf{x}_1,\mathbf{0}) + \cdots + \alpha_n(\mathbf{x}_n,\mathbf{0}) + \beta_1(\mathbf{0},\mathbf{y}_1) + \cdots + \beta_m(\mathbf{0},\mathbf{y}_m) = (\mathbf{x},\mathbf{y}).$$

Hence the vectors are spanning and therefore a basis for $X \oplus Y$. □

(c) Let $(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2) \in X \oplus Y$ and let α and β be scalars. Then

$$\begin{aligned}T\left(\alpha(\mathbf{x}_1,\mathbf{y}_1) + \beta(\mathbf{x}_2,\mathbf{y}_2)\right) &= T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2)\\ &= (\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) - (\alpha\mathbf{y}_1 + \beta\mathbf{y}_2)\\ &= \alpha(\mathbf{x}_1 - \mathbf{y}_1) + \beta(\mathbf{x}_2 - \mathbf{y}_2) = \alpha T(\mathbf{x}_1,\mathbf{y}_1) + \beta T(\mathbf{x}_2,\mathbf{y}_2).\end{aligned}$$

Thus T is linear. □

(d) Let $\mathbf{v} \in X + Y$. Then there exists $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ such that $\mathbf{v} = \mathbf{x} + \mathbf{y}$. But then $T(\mathbf{x},-\mathbf{y}) = \mathbf{x} - (-\mathbf{y}) = \mathbf{x} + \mathbf{y} = \mathbf{v}$. Thus $X + Y \subset \mathrm{Ran}(T)$. On the other hand, for any $(\mathbf{x},\mathbf{y}) \in X \oplus Y$, we have $T(\mathbf{x},\mathbf{y}) = \mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) \in X + Y$. So the other inclusion holds and we therefore have $X + Y = \mathrm{Ran}(T)$. □

(e) Suppose $(\mathbf{x},\mathbf{y}) \in \mathrm{Ker}(T)$. Then $\mathbf{0} = T(\mathbf{x},\mathbf{y}) = \mathbf{x} - \mathbf{y}$, which implies $\mathbf{x} = \mathbf{y}$. Since $\mathbf{x} \in X$ and $\mathbf{y} \in Y$, their being equal implies $\mathbf{x} = \mathbf{y} \in X \cap Y$. Thus $\mathrm{Ker}(T) = \{(\mathbf{w},\mathbf{w})\colon \mathbf{w} \in X \cap Y\}$. We claim that

$$\mathrm{Ker}(T) \ni (\mathbf{w},\mathbf{w}) \mapsto \mathbf{w} \in X \cap Y$$

defines an isomorphism. Indeed, it is clearly linear and its inverse is simply the map

$$X \cap Y \ni \mathbf{w} \mapsto (\mathbf{w},\mathbf{w}) \in \mathrm{Ker}(T).$$

Thus $\mathrm{Ker}(T) \cong X \cap Y$ as claimed. □

(f) The rank-nullity theorem for T states

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(X \oplus Y).$$

By part (b), the right-hand side is $\dim(X) + \dim(Y)$. By part (d), $\mathrm{rank}(T) = \dim(\mathrm{Ran}(T)) = \dim(X + Y)$. By part (e), $\mathrm{nullity}(T) = \dim(\mathrm{Ker}(T)) = \dim(X \cap Y)$. Substituting all of this into the above equation yields

$$\dim(X + Y) + \dim(X \cap Y) = \dim(X) + \dim(Y).$$

Subtracting $\dim(X \cap Y)$ from each side yields the desired equation. □

4. $(\Longrightarrow)$: Suppose $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{F}^m$. By an observation from lecture, this means the RREF of A has a pivot in every row and therefore $\text{rank}(A) = m$. By the matrix version of the rank-nullity theorem we have $\text{nullity}(A^T) = m - \text{rank}(A) = 0$. Thus $\text{Ker}(A^T)$ is the zero subspace, which only contains $\mathbf{0}$. Hence $A^T\mathbf{x} = \mathbf{0}$ has a unique solutions (namely $\mathbf{x} = \mathbf{0}$).

   $(\Longleftarrow)$: Suppose $A^T\mathbf{x} = \mathbf{0}$ has a unique solution. By an observation from lecture, this means the RREF of A^T has a pivot in every column. Since $A^T \in M_{n\times m}$, this means $\text{rank}(A^T) = m$. But the rank of A and A^T agree, so $\text{rank}(A) = m$. Thus the RREF of A has a pivot in every row, and therefore $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{F}^m$. $\square$

5. We let A be the matrix with rows $\mathbf{v}_1^T, \mathbf{v}_2^T, \mathbf{v}_3^T$. Then its RREF is:

$$\begin{pmatrix} 1 & 0 & 2 & 0 & \frac{7}{2} \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}.$$

   Columns 3 and 5 are missing pivots. So by the algorithm discussed in lecture, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_3, \mathbf{e}_5$ form a basis for $\mathbb{R}^5$. (Note that we can verify this by checking that the RREF of the matrix $(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{e}_3\mathbf{e}_5)$ is I_5.)