§3 Relations

Def A relation on a set A is a subset $C \subset A \times A$. For $x, y \in A$, we write $x C y$ to mean $(x, y) \in C$.

Ex Let $f: \mathbb{R} \to \mathbb{R}$ be a function. The graph of f

$$G := \{(x, f(x)) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

is a relation on $\mathbb{R}$: $x G y$ iff $y = f(x)$. Note that since f is a rule of assignment (see §1), for a single $x \in \mathbb{R}$ we cannot have $x G y$ and $x G z$ for $y \neq z$ (i.e. f is not multi-valued), but this need not be true for arbitrary relations.

- General relations are not so useful due to their lack of structure, so it is common to study relations that have properties imposed on them. In this section we will focus on two: equivalence relations and order relations.

Equivalence Relations

Def An equivalence relation on a set A is a relation C on A satisfying

1. $x C x$ for all $x \in A$ (Reflexivity)
2. If $x C y$, then $y C x$ (Symmetry)
3. If $x C y$ and $y C z$, then $x C z$ (Transitivity)

In this case, one typically writes $x \sim y$ to denote $x C y$.

Ex ① Define a relation C on $\mathbb{R}$ by $x C y$ iff $(x-y) \in \mathbb{Q}$. This is an equivalence relation:

Reflexivity: $x - x = 0 \in \mathbb{Q}$ so $x C x$

Symmetry: If $x C y$ then $x - y \in \mathbb{Q} \Rightarrow (y-x) = -(x-y) \in \mathbb{Q}$, so $y \mathbb{Q} x$.

Transitivity: If $x C y$ and $y C z$, then $(x-y), (y-z) \in \mathbb{Q}$, and so

$$x - z = x - y + y - z = (x-y) + (y-z) \in \mathbb{Q}$$

hence $x C z$.

② Define a relation C on $\mathbb{R}$ by $x C y$ iff $(x-y) \in \mathbb{R} \setminus \mathbb{Q}$. This is not an equivalence relation. Which conditions does it fail? ①, ③

③ Define a relation B on the set of all human beings who were born by aBb iff a and b were born in the same year. This an equivalence relation. $\square$

Def For an equivalence relation $\sim$ on a set A, the equivalence class of $x \in A$ is the set

$$[x] := \{ y \in A : x \sim y \}$$

Ex For two Cauchy sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$, define $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ iff

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

(Check this is an equivalence relation) Equivalently, $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ iff $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Thus we can identify $[(x_n)_{n\in\mathbb{N}}]$ with its limit (which need not be in $\mathbb{Q}$). The resulting collection of equivalence classes can then be identified with $\mathbb{R}$. In fact, this is one way to construct $\mathbb{R}$ from $\mathbb{Q}$. $\square$

Lemma For an equivalence relation $\sim$ on a set A and $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \phi$.

Proof First suppose $x \sim y$. Then $\forall z \in [y]$ we have $y \sim z$ and hence $x \sim z$ by transitivity. Thus $z \in [x]$ and so $[y] \subseteq [x]$. Using symmetry we can run the same argument with x and y swapped, giving $[x] \subseteq [y]$ and so $[x] = [y]$.
So if $[x] \neq [y]$, the above tells us we do not have $x \sim y$. Suppose, towards a contradiction that $\exists z \in [x] \cap [y]$. Then $x \sim z$ and $y \sim z$. Using symmetry, $z \sim y$ and by transitivity $x \sim y$, a contradiction. Thus $[x] \cap [y] = \phi$ when $[x] \neq [y]$. $\square$

- The above lemma tells us that the equivalence classes divide the set A up into disjoint (nonempty) subsets.

Def A <u>partition</u> of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

Ex Fix $n \in \mathbb{N}$ and for $x, y \in \mathbb{Z}$, write $x \sim y$ iff $x - y$ is divisible by n. That is, $x \sim y$ iff $x \pmod{n} = y \pmod{n}$. So

$$\mathbb{Z} = [0] \cup [1] \cup \cdots \cup [n-1]$$

and $[j]$ consists of all $x \in \mathbb{Z}$ whose remainder is j when divided by n. $\square$

Order Relations

Def An <u>order relation</u> on a set A is a relation C on A satisfying

1. For $x, y \in A$ with $x \neq y$, either $x C y$ or $y C x$. (<u>Comparability</u>)
2. For no $x \in A$ does $x C x$ hold. (<u>Nonreflexivity</u>)
3. If $x C y$ and $y C z$, then $x C z$. (<u>Transitivity</u>)

In this case, one typically writes $x < y$ for $x C y$. For $x, y \in A$ with $x < y$, we denote an <u>interval</u> by
$$(x, y) := \{ a \in A : x < a < y \}$$
"$x < a$ and $a < b$"

We write $x \leq y$ to mean either $x < y$ or $x = y$.

EX ① On $\mathbb{R}$ define "$<$" in the usual way ($x < y$ iff y is strictly greater than zero). Then $<$ is an order relation:

comparability, reflexivity: these are essentially axioms on $\mathbb{R}$, but can be confirmed by plotting on a number line

transitivity: if $x < y$ and $y < z$, then $x < z$

② On $\mathbb{R}$, the relation $x \leq y$ is <u>not</u> an order relation. Which properties does it fail to satisfy? (But it <u>is</u> a "partial order relation".)

③ On $\mathbb{R}$, define a relation C by $x C y$ if $|x| < |y|$. This is <u>not</u> an order relation (why?), but the relation D defined by $x D y$ if $|x| < |y|$, or if $|x| = |y|$ and $x < y$, is (check this) $\square$

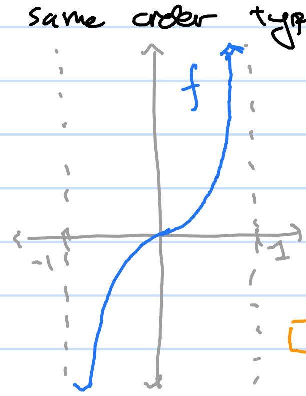
Def If A and B are sets with order relations $<_A$ and $<_B$, respectively, we say they have the same <u>order type</u> if there is a bijection $f : A \to B$ such that
$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2)$$
(Exercise: show that in this case $b_1 <_B b_2 \Rightarrow f^{-1}(b_1) <_B f^{-1}(b_2)$.)

EX $(-1, 1)$ and $\mathbb{R}$ with their usual orderings have the same order type. Consider $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \frac{x}{1 - x^2}$.

Then $f'(x) = \frac{x^2 + 1}{(1 - x^2)^2}$ and $f'(x) > 0$ for all $x \in (-1, 1)$.

Consequently $x < y \Rightarrow f(x) < f(y)$ and f is injective.

Since
$$\lim_{x \nearrow 1} f(x) = +\infty \qquad \lim_{x \searrow -1} f(x) = -\infty$$
the intermediate value theorem implies f is surjective. $\square$



Def Let A be a set with order relation $<$. For a subset $A_0 \subset A$, we say $x_0 \in A_0$ is the
- largest element in A_0 if $a \le x_0$ for all $a \in A_0$.
- smallest element in A_0 if $x_0 \le a$ for all $a \in A_0$.

We say $x_0 \in A$ is a(n)
- upper bound for A_0 if $a \le x_0$ for all $a \in A_0$, and we say A_0 is bounded above
- lower bound for A_0 if $x_0 \le a$ for all $a \in A_0$, and we say A_0 is bounded below

If the set of all upper bounds of A_0 has a smallest element, that element is called the least upper bound (or supremum) of A_0. If the set of all lower bounds of A_0 has a largest element, that element is called the greatest lower bound (or infimum) of A_0.

Ex ① Give $\mathbb{R}$ the usual ordering. The intervals

$$(-1,2) \qquad [-1,2) \qquad (-1,2] \qquad [-1,2]$$

all have least upper bounds of 2 and greatest lower bounds of -1.
The ray $(-1,\infty)$ has no upper bounds and hence no least upper bd.

② Give $\mathbb{Q}$ the usual ordering. The set $(-\infty,\sqrt{2}) \cap \mathbb{Q}$ has no least upper bound in $\mathbb{Q}$: if $q \in \mathbb{Q}$ is any upper bound, then $\sqrt{2} < q$. We can then find $r \in \mathbb{Q}$ with $\sqrt{2} < r < q$ (by the density of $\mathbb{Q}$). Consequently the set of upper bounds does not have a smallest element. ($\sqrt{2}$ wants to be the l.u.b., but can't be since $\sqrt{2} \notin \mathbb{Q}$). □

Def An ordered set A is said to have the least upper bound property if every nonempty subset $A_0 \subset A$ that is bounded above has a least upper bound. A is said to have the greatest lower bound property if every nonempty subset $A_0 \subset A$ that is bounded below has a greatest lower bound. (Exercise: show these properties are equivalent).

Ex $\mathbb{R}$ has the least upper bound property while $\mathbb{Q}$ does not. □

Partial Order (Relations)

Def A partial order (relation) on a set A is a relation C on A satisfying
① xCx for all $x \in A$ (Reflexivity)
② If xCy and yCx, then $x=y$. (Antisymmetry)
③ If xCy and yCz, then xCz. (Transitivity)

In this case, we typically write $x \preceq y$ for xCy.

Ex ① Define a relation on $\mathbb{R}$ by $x \preceq y$ iff $x < y$ or $x = y$, where $<$ is the usual order relation on $\mathbb{R}$. Then $\preceq$ is a partial order.

② Let $\mathcal{C}$ be the collection of subsets $A \subset \mathbb{R}$. For $A, B \subseteq \mathbb{R}$, $A \preceq B$ iff $A \subset B$ is a partial order on $\mathcal{C}$.

Reflexivity: $A \subset A$ so $A \preceq A$.

Antisymmetry: If $A \preceq B$ and $B \preceq A$, then $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

Transitivity: If $A \preceq B$ and $B \preceq C$, then $A \subseteq B \subseteq C \Rightarrow A \subseteq C \Rightarrow A \preceq C$.

③ Let $\mathcal{C}$ be as above. For $A, B \subseteq \mathbb{R}$, $A \preceq B$ iff $A \supset B$ is a partial order on $\mathcal{C}$. (Exercise: check this). □

- Note that in the definition of a partial order, it may be that for $x, y \in A$ neither $x \preceq y$ nor $y \preceq x$ holds; that is, not all elements are comparable. For example, $[0,1], [2,3] \subseteq \mathbb{R}$ are not comparable under either of the partial orderings in ②, ③.

Lemma If A is a set with order relation $<$, then the relation $x \preceq y$ iff $x < y$ or $x = y$ is a partial order relation.

Proof We must check the three properties in the definition of a partial order are satisfied: ① Reflexivity, ② Antisymmetry, and ③ Transitivity.

①: Given $a \in A$, we of course have $a = a$, and so $a \preceq a$.

②: Let $a, b \in A$ satisfy $a \preceq b$ and $b \preceq a$. Suppose, towards a contradiction, that $a \neq b$. Then $a \preceq b$ must mean $a < b$ and $b \preceq a$ must mean $b < a$. So $a < b$ and $b < a$ and transitivity implies $a < a$. But this contradicts nonrelexivity ② in the definition of an order relation.

③: Let $a, b, c \in A$ satisfy $a \preceq b$ and $b \preceq c$. If $a = b$ or $b = c$, then $a \preceq c$ follows after the appropriate substitution. If $a \neq b$ and $b \neq c$, then $a \preceq b \Rightarrow a < b$ and $b \preceq c \Rightarrow b < c$. We then have $a < c \Rightarrow a \preceq c$ by transitivity of $<$. □

- Suppose $\preceq$ is a partial order on a set A. Is $x C y$ iff $x \preceq y$ and $x \neq y$ an order relation?

(Equivalently, if $\preceq$ corresponds to a subset $D \subseteq A \times A$, then $C := \{(x,y) \in D \mid x \neq y\}$.) No, because we can have $x, y \in A$ with $x \neq y$ but neither $x C y$ nor $y C x$ holds.

§7 Countable and Uncountable Sets

Notation $\mathbb{N} := \mathbb{Z}_+$.

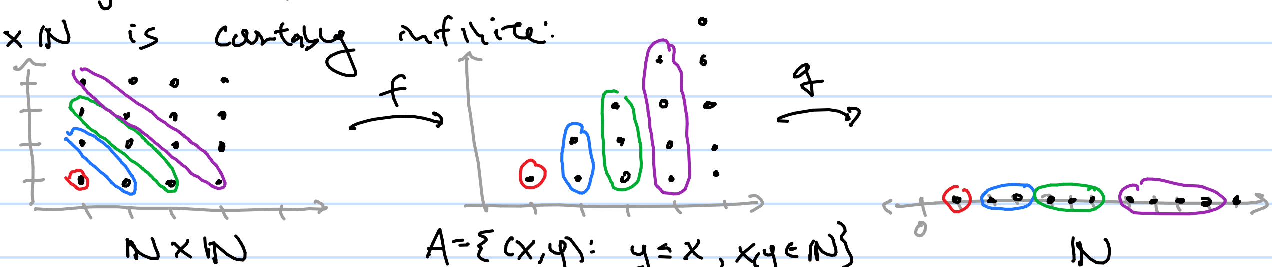
- In §6, a set A was defined to be finite if for some $n \in \mathbb{N}$, there is a bijection $f: A \to \{1, 2, \ldots, n\}$ (or if $A = \emptyset$).

Def We say a set A is infinite if it is not finite. We say A is countably infinite if there is a bijection $f: A \to \mathbb{N}$. We say A is countable if it is either finite or countably infinite, and otherwise say it is uncountable.

- It was also shown in §6 that $\mathbb{N}$ is not finite. Consequently $\mathbb{N}$ is infinite and so is any countably infinite set.

Ex 1) $\mathbb{Z}$ is countably infinite:
$$f(n) := \begin{cases} 2n & \text{if } n > 0 \\ -2n+1 & \text{if } n \leq 0 \end{cases}$$
is a bijection $f: \mathbb{Z} \to \mathbb{N}$.

2) $\mathbb{N} \times \mathbb{N}$ is countably infinite:



$$f(x,y) := (x+y-1,\ y) \qquad g(x,y) = \underbrace{\frac{x(x-1)}{2}}_{= \sum_{j=1}^{x-1} j} + y$$

3) $[0,1]$ is uncountable. Suppose, towards a contradiction that it is countable. Since $\mathbb{N} \ni n \mapsto \frac{1}{n} \in [0,1]$ is an injection, it cannot be finite, so we just need to rule out it being countably infinite. If $f: [0,1] \to \mathbb{N}$ is a bijection, let $x_n \in [0,1)$ be the unique element s.t. $f(x_n) = n$. Now each x_n has a decimal expansion
$$x_n = a_0^{(n)} . a_1^{(n)} a_2^{(n)} a_3^{(n)} \ldots$$
with $a_j^{(n)} \in \{0, 1, \ldots, 9\}$. This expansion is unique provided we exclude infinite repetitions of 9's. For each $n \in \mathbb{N}$, set

$b_n := \min \{0,1\} \setminus \{a_n^{(n)}\}$. That is, b_n is either 0 or 1 and is different from $a_n^{(n)}$ (the nth digit of x_n). Define $x \in [0,1]$ by the decimal expansion

$$x := 0.b_1 b_2 b_3 \cdots$$

Since $x \in [0,1]$, $f(x) = n$ for some $n \in \mathbb{N}$. Since f is a bijection, we must have $x = x_n$, but the nth digits are $b_n \neq a_n^{(n)} \Rightarrow x \neq x_n$, a contradiction. $\square$

Lemma Every subset $A \subseteq \mathbb{N}$ is countable.

Proof If A is finite then it is countable by definition. So assume A is infinite. Now, define a map $h: \mathbb{N} \to A$ by letting $h(1)$ be the smallest element of A and then defining $h(n)$ recursively as the smallest element of

$$A \setminus \{h(1), h(2), \ldots, h(n-1)\}.$$

Note that the above set cannot be empty since this would imply $A \subseteq \{h(1), \ldots, h(n-1)\}$, which contradicts A being infinite. Thus h is well-defined.

h is also injective: if $n \neq m$ then either $n < m$ and

$$h(n) \in \{h(1), \ldots, h(m-1)\} \not\ni h(m)$$

or $m < n$ and

$$h(m) \in \{h(1), \ldots, h(n-1)\} \not\ni h(n)$$

To see that h is surjective, fix $a_0 \in A$.

Claim: $\exists\, n \in \mathbb{N}$ such that $a_0 < h(n)$

Indeed, if not then $h(n) \leq a_0$ $\forall n \in \mathbb{N}$ and so $h(\mathbb{N}) \subseteq \{1, \ldots, a_0\}$. This implies $h(\mathbb{N})$ is finite, but $\mathbb{N}$ is infinite and h is injective so this is a contradiction (by the pigeonhole principle). $\square$

Let $n \in \mathbb{N}$ be as in the claim. Since $a_0 < h(n)$, it must be that

$$a_0 \notin A \setminus \{h(1), \ldots, h(n-1)\}.$$

Since $a_0 \in A$, this is only possible if $a_0 \in \{h(1), \ldots, h(n-1)\}$. Hence $\exists\, 1 \leq j \leq n-1$ such that $h(j) = a_0$. Thus h is surjective. We have therefore constructed a bijection $h: \mathbb{N} \to B$ and hence B is countably infinite. $\square$

Thm Let B be a non-empty set. The following are equivalent:

i) B is countable.

ii) There exists a surjective function $f: \mathbb{N} \to B$.

iii) There exists an injective function $g: B \to \mathbb{N}$.

Proof

(i ⇒ ii): Suppose B is countable. If B is finite, then for some $n \in \mathbb{N}$ we have a bijection $h: B \to \{1, 2, \dots, n\}$. Define $f: \mathbb{N} \to B$ by

$$f(k) := \begin{cases} h^{-1}(k) & \text{if } 1 \le k \le n \\ h^{-1}(1) & \text{otherwise.} \end{cases}$$

Then $f(\{1, \dots, n\}) = h^{-1}(\{1, \dots, n\}) = B$, so f is surjective. If B is countably infinite, then by definition there exists a bijection $h: B \to \mathbb{N}$. Set $f := h^{-1}$, so that $f: \mathbb{N} \to B$ is a bijection and in particular is surjective.

(ii ⇒ iii): Let $f: \mathbb{N} \to B$ be a surjection. Define $g: B \to \mathbb{N}$ by

$$g(b) = \min\{n \in \mathbb{N} \mid f(n) = b\}.$$

Since f is surjective, $\{n \in \mathbb{N} \mid f(n) = b\} \neq \emptyset$ for all $b \in B$ and so g is well-defined. To see that g is injective, suppose $g(b_1) = n = g(b_2)$ for some $n \in \mathbb{N}$. Then by definition of g, we have $b_1 = f(n) = b_2$. So g is injective.

(iii ⇒ i): Suppose $g: B \to \mathbb{N}$ is injective. Then $g: B \to g(B)$ is a bijection and by the previous lemma $g(B)$ is countable. Hence B is countable. $\square$

Cor A subset of a countable set is countable.

Proof If $A \subset B$ and B is countable, then by the theorem there is an injection $g: B \to \mathbb{N}$. The function $g|_A: A \to \mathbb{N}$ is also an injection and hence A is countable. $\square$

Cor $\mathbb{Q}$ is countable

Proof Recall that $\mathbb{N} \times \mathbb{N}$ is countable, so the theorem implies there exists a surjection $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Define $h: \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ by

$$h(a, b) := \begin{cases} \frac{a}{2b} & \text{if } a \text{ even} \\ 0 & \text{if } a = 1 \\ -\frac{(a-1)}{2b} & \text{if } a \text{ odd} \end{cases}$$

Since $\mathbb{Q} = \{\frac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\}$, h is surjective: $h(2n, m) = \frac{n}{m}$ for $n > 0$, $h(1, m) = 0 = \frac{n}{m}$ for $n = 0$, and $h(-2n+1, m) = \frac{n}{m}$ for $n < 0$. Thus $h \circ f: \mathbb{N} \to \mathbb{Q}$ is a surjection and so $\mathbb{Q}$ is countable. $\square$

Thm A countable union of countable sets is countable.

Proof Let I be a countable set and for each $i \in I$ let A_i be a countable set. We must show $\bigcup_{i \in I} A_i$ is countable. For each $i \in I$, let $g_i : \mathbb{N} \to A_i$ be a surjection, and let $h : \mathbb{N} \to I$ be a surjection. Then $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{i \in I} A_i$ defined by $f(a,b) := g_{h(a)}(b)$ is a surjection. Precomposing f with a surjection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ shows the union is countable. $\square$

Thm A finite product of countable sets is countable.

Proof For $m \in \mathbb{N}$, let $A_1, \ldots, A_m$ be countable sets. We must show $A_1 \times A_2 \times \cdots \times A_m$ is countable. We proceed by induction on m. For $m = 1$, we have A_1 is countable by assumption and we are done. For the induction step, suppose we have shown $A_1 \times \cdots \times A_{m-1}$ is countable. Then there exists a surjection $f_1 : \mathbb{N} \to A_1 \times \cdots \times A_{m-1}$. Since A_m is countable, we can also find a surjection $f_2 : \mathbb{N} \to A_m$. Consequently, the map

$$\mathbb{N} \times \mathbb{N} \ni (a,b) \mapsto (f_1(a), f_2(b)) \in (A_1 \times \cdots \times A_{m-1}) \times A_m = A_1 \times \cdots \times A_m$$

is a surjection. Precomposing with a surjection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ shows $A_1 \times \cdots \times A_m$ is countable. Induction then completes the proof. $\square$

Def Given a set A, its <u>power set</u> is the collection of all subsets of A and is denoted $\mathcal{P}(A)$.

- $\mathcal{P}(\emptyset) = \emptyset$, but for $A \neq \emptyset$ $\mathcal{P}$ is always larger:

$$\mathcal{P}(\underbrace{\{0,1\}}_{\text{2-elements}}) = \{\underbrace{\emptyset, \{0\}, \{1\}, \{0,1\}}_{\text{4-elements}}\}$$

Thm Let A be a nonempty set. There is no surjective map $f : A \to \mathcal{P}(A)$ and no injective map $g : \mathcal{P}(A) \to A$.

Proof Suppose, towards a contradiction, that $f : A \to \mathcal{P}(A)$ is a surjection. Then for each $a \in A$, $f(a)$ is a subset of A. Hence we can ask whether or not $a \in f(a)$. Define $B \subset A$ by

$$B := \{ a \in A : a \notin f(a) \}.$$

Now, since f is surjective and $B \subset A$, $\exists\, b \in A$ such that $f(b) = B$. So, is $b \in B$? If yes, then $b \notin f(b) = B$ a contradiction. If no, then $b \notin B = f(b)$ which implies $b \in B$ by definition of B, again a contradiction. Thus no matter what we arrive at a contradiction and hence f cannot exist.

If $g : \mathcal{P}(A) \to A$ is a injection, then define $f : A \to \mathcal{P}(A)$ by $f = g^{-1}$ on $g(\mathcal{P}(A))$ and arbitrarily elsewhere. Then f is a surjection, which contradicts the above. $\square$

§9 Axiom of Choice

- In order to motivate the axiom of choice, we will begin by "proving" the following characterization of infinite sets:

Thm Let A be a set. The following are equivalent:

(i) A is infinite.

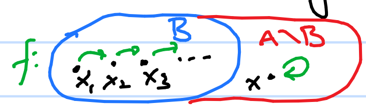
(ii) There exists an injective function $f: \mathbb{N} \to A$.

(iii) There is a bijection of A with a proper subset of itself.

Proof

(ii $\Rightarrow$ iii): Denote $B := f(\mathbb{N})$ and $x_n := f(n)$ for $n \in \mathbb{N}$. Define $g: A \to A \setminus \{x_1\}$ by

$$g(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$



Then g is a bijection.

(iii $\Rightarrow$ i): Suppose $B \subsetneq A$ and $f: A \to B$ is a bijection. If A is finite, then for some $n \in \mathbb{N}$, there exists a bijection $g: A \to \{1, .., n\}$. But then $g \circ f^{-1}: B \to \{1, .., n\}$ is a bijection and A and B both contain n elements. This contradicts $B \subsetneq A$. (see also §6).

(i $\Rightarrow$ ii): Since A is infinite, it is non-empty. So let $f(1) \in A$ be arbitrary, and inductively define $f: \mathbb{N} \to A$ by choosing

$$f(n) \in A \setminus \{f(1), .., f(n-1)\}.$$

Note $A \setminus \{f(1), .., f(n-1)\}$ must be non-empty for all $n \in \mathbb{N}$, since otherwise we would contradict A being infinite. Hence f is well-defined. It is injective since $n \neq m$ implies

$$f(m) \notin \{f(1), .., f(m-1)\} \ni f(n).$$

$\square$

- There is an issue with the above proof:

  Our definition of f is ambiguous. How do we "choose" an element of $A \setminus \{f(1), .., f(n-1)\}$? How do we know we can do this for all $n \in \mathbb{N}$?

  It is very non-trivial, but the set theoretic operations we have used so far are not enough to overcome this issue. Consequently, we require the following patch:

Axiom of Choice Given a collection $\mathcal{A}$ of disjoint non-empty sets, there exists a set C such that for all $A \in \mathcal{A}$, $C \cap A$ consists of a single element.

$\boxed{\text{Lemma}}$ Given a collection $\mathcal{B}$ of non-empty sets (not necessarily disjoint), there exists a function

$$c: \mathcal{B} \longrightarrow \bigcup_{B \in \mathcal{B}} B$$

such that $c(B) \in B$ for all $B \in \mathcal{B}$. (c is called a choice function.)

Proof Since the sets $\mathcal{B}$ are not assumed to be disjoint, we cannot immediately invoke the axiom of choice. To get disjoint collection, we do the following trick: for $B \in \mathcal{B}$, define

$$\tilde{B} := \{ (B, x) : x \in B \} \subseteq \mathcal{B} \times \bigcup_{B \in \mathcal{B}} B.$$

Note that since B is non-empty, $\exists x \in B$ and so $(B, x) \in \tilde{B} \neq \phi$. Now, if $B_1, B_2 \in \mathcal{B}$ are distinct sets, then $\tilde{B}_1 \cap \tilde{B}_2 = \phi$ since $(B_1, x) \neq (B_2, y)$ for any $x \in B_1$ and $y \in B_2$. Hence $\mathcal{A} := \{ \tilde{B} \mid B \in \mathcal{B} \}$ is a collection of disjoint non-empty sets. Let C be as in the axiom choice. Then $\forall \tilde{B} \in \mathcal{A}$, $C \cap \tilde{B}$ consists of a single element (B, x). Denote $c(B) := x$. Then c is the desired choice function. $\square$

- With this in hand we can fix the proof of the theorem above: let $\mathcal{B}$ denote the collection of non-empty subsets of A and let $c: \mathcal{B} \to \bigcup_{B \in \mathcal{B}} B$ be a choice function. Define $f(1) := c(A)$ and inductively define

$$f(n) := c(A \setminus \{ f(1), \ldots, f(n-1) \}).$$

Since $f(n) \in A \setminus \{ f(1), \ldots, f(n-1) \}$ by definition of a choice function, we have $f(n) \neq f(j)$ for each $j < n$. That is, $f: \mathbb{N} \to A$ is injective.

§ 10 Well-Ordered Sets

Def A set A with order relation $<$ is well-ordered if every non-empty subset has a smallest element.

Ex ① $\mathbb{N}$ with its usual order is well-ordered

② $\mathbb{Z}$ with its usual order is not well-ordered: $\{x \in \mathbb{Z} \mid x < 0\}$ has no smallest element.

Define a relation C on $\mathbb{Z}$ by $x C y$ iff $|x| < |y|$ or if $|x| = |y|$ and $x < y$.
Then this is an order relation making $\mathbb{Z}$ well-ordered (Exercise). □

Def Let A and B be sets with respective order relations $<_A$ and $<_B$.
The order relation $<$ on $A \times B$ defined by

$$(a_1, b_1) < (a_2, b_2) \text{ iff } a_1 <_A a_2 \text{ or if } a_1 = a_2 \text{ and } b_1 <_B b_2$$

is called the dictionary order relation on $A \times B$.

Prop Let A and B be sets with respective order relations $<_A$ and $<_B$ making them well-ordered. Then $A \times B$ with the dictionary order relation is well-ordered.

Proof Let $X \subseteq A \times B$ be non-empty. Let $Y := \{a \in A \mid (a, b) \in X \text{ for some } b \in B\}$.
Then $Y \subset A$ is non-empty and hence has a smallest element $a_0 \in Y$. Define $Z := \{b \in B : (a_0, b) \in X\}$. Since $a_0 \in Y$, $\exists\, b \in B$ such that $(a_0, b) \in X$ and so $b \in Z \neq \phi$. Thus $Z \subset B$ has a smallest element b_0. Then $(a_0, b_0) \in X$ is the smallest element. □

- Using the above proposition, we see that $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times (\mathbb{N} \times \mathbb{N})$, etc. are all well-ordered with the dictionary order relation. What about $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots$?

Ex Let $\mathbb{N}^\omega$ denote the set of sequences $(a_n)_{n \in \mathbb{N}}$ with $a_n \in \mathbb{N}$. Define an order relation $\mathbb{N}^\omega$ by

$$(a_n)_{n \in \mathbb{N}} < (b_n)_{n \in \mathbb{N}} \text{ iff for some } n \geq 1,\ a_i = b_i \text{ for } i < n, \text{ and } a_n < b_n.$$

Then $\mathbb{N}^\omega$ is not well-ordered:

$$X := \{(a_n)_{n \in \mathbb{N}} : a_n \in \{1, 2\} \text{ for all } n \in \mathbb{N} \text{ and } a_n = 2 \text{ for exactly one } n \in \mathbb{N}\}.$$

then for $(a_n), (b_n) \in X$ we have $(a_n) < (b_n)$ provided the 2 occurs later in the first sequence. Consequently X has no smallest element. □

- Note that $\mathbb{N}^\omega$ is uncountable (by the same diagonal argument which showed $[0,1]$

is uncountable). Perhaps this is an obstruction and $\mathbb{N}^\omega$ has no order relation making it well-ordered? For example, can $\mathbb{R}$ be well-ordered? It turns out that by using the axiom of choice, any set admits an order relation that makes it well-ordered.

Well-Ordering Theorem (Zermelo 1904)
For any set A there exists an order relation on A that is a well-ordering.

- We won't prove this in this course, but we note that this theorem is actually equivalent to the axiom of choice (and another theorem we'll discuss in the next section). Unfortunately, the ~~proof~~ is not constructive, so we cannot write down, for example, an order relation on $\mathbb{R}$ that makes it a well-ordering. We just know one exists.

Cor There exists an uncountable well-ordered set.

- An important example for when we start discussing "topology" comes from the set in the above corollary. We will define and examine this example shortly.

Def Let X be a well-ordered set. Given $\alpha \in X$, the section of X by α is

$$S_\alpha := \{x \in X \mid x < \alpha\}.$$

Lemma There exists a well-ordered set A with largest element Ω such that the section S_Ω of A by Ω is uncountable but every other section is countable.

Proof Let B be an uncountable well-ordered set. Define $C := \{1,2\} \times B$ and equip it with the dictionary relation (so it is also well-ordered). Note C has sections which are uncountable: $\forall b \in B$ the section of C by $(2,b)$ is uncountable since it contains $\{1\} \times B$. Let Ω be the smallest element of

$$D := \{c \in C \mid S_c \text{ is uncountable}\},$$

which exists since C is well-ordered. Define $A := S_\Omega \cup \{\Omega\}$. Then $a < \Omega$ for all $a \in A \setminus \{\Omega\}$ by definition of S_Ω, which means Ω is the largest element of A. The section of A by Ω is precisely S_Ω (the section of C by Ω) and is therefore uncountable since $\Omega \in D$. For $a \in A \setminus \{\Omega\}$ we must have S_a is countable since otherwise $a \in D$ and $\Omega \le a$, a contradiction. $\square$

Def S_Ω as in the proof of the previous lemma is called a minimal uncountable well-ordered set. We denote $\bar{S}_\Omega := S_\Omega \cup \{\Omega\}$ $(=A)$.

Thm If $A \subset S_\Omega$ is countable, then A has an upper bound in S_Ω.

Proof For all $a \in A$, S_a is countable and hence

$$B := \bigcup_{a \in A} S_a$$

is countable. Consequently $B \subsetneq S_\Omega$. Let $x \in S_\Omega \setminus B$. If $x < a$ for some $a \in A$, then $x \in S_a \subset B$, a contradiction. Thus $x \geq a$ for all $a \in A$; that is, x is an upper bound for A. $\square$

§11 The Maximum Principle

Def For a set A, a relation C on A is a strict partial order relation if it satisfies:

1. For no $a \in A$ does aCa hold. (Nonreflexivity)
2. If xCy and yCz, then xCz. (Transitivity)

In this case, one typically writes $x \prec y$ for xCy. We also write $x \preceq y$ if $x = y$ or $x \prec y$.

- This is just the definition of an order relation without comparability. Hence every order relation is also a strict partial order relation. The converse is not true (see the example below), but a set A with strict partial order relation $\prec$ could have a subset $B \subset A$ on which $\prec$ is an order relation.

Def Let A be a set with a strict partial order relation $\prec$. We say a subset $B \subset A$ is totally (or simply) ordered if for every $x, y \in B$ with $x \neq y$, either $x \prec y$ or $y \prec x$. (B is also called a chain.)

Ex Let $\mathcal{C}$ be the collection of subsets of $\mathbb{N}$. For $A, B \subseteq \mathbb{N}$, write $A \prec B$ iff $A \subsetneq B$. Then $\prec$ is a strict partial order relation. Note that neither $\{1\} \prec \{2\}$ nor $\{2\} \prec \{1\}$ holds, so $\prec$ is not an order relation. However, the subcollection

$$\mathcal{B} = \{ \{1, 2, \ldots, n\} : n \in \mathbb{N} \}$$

is totally ordered. $\square$

- If $\prec$ is a strict partial order, then the relation $x \preceq y$ iff $x \prec y$ or $x = y$ is a partial order relation. Conversely if $\preceq$ is a partial order, then $x \prec y$ iff $x \preceq y$ and $x \neq y$ is a strict partial order. (Exercise check this.) (Consequently, the content of this section can be (and often is) expressed equivalently using partial orders.

Thm (Maximum Principle) Let A be a set with a strict partial order relation. Then there exists a maximal totally ordered subset $B \subset A$.

- Here "maximal" means if $C \subset A$ is totally ordered and $C \supset B$, then $C = B$.

Is the collection $\mathcal{B}$ in the above example maximal? No, but $\mathcal{B} \cup \{\mathbb{N}\}$ is.

- We will only sketch the proof of the Maximum Principle:

Proof (sketch) Let $\prec$ be the strict partial order on A. By the well-ordering theorem, there exists an order relation $<_0$ on A making it well-ordered. Let a_1 be the smallest element of $(A, <_0)$. Consider the set
$$\{a \in A \setminus \{a_1\} \mid \text{either } a \prec a_1 \text{ or } a_1 \prec a\}.$$
If the set is empty, then $\{a_1\}$ is a maximal totally ordered set. Otherwise, let a_2 be the smallest element of this set (with respect to $<_0$). Then $\{a_1, a_2\}$ is totally ordered. Then consider
$$\{a \in A \setminus \{a_1\} \mid a \text{ is comparable to } a_1, a_2\}.$$
If it is empty, then $\{a_1, a_2\}$ is our maximal totally ordered set. Otherwise, let a_3 be the smallest element of this set (w.r.t. $<_0$). Then $\{a_1, a_2, a_3\}$ is totally ordered. Regular induction will give us the desired set if A is countable, otherwise we require "induction for well-ordered sets" which is not so hard to deduce. $\square$

- From the Maximum Principle, one can deduce the famous Zorn's Lemma. We prove it below in the form presented by Munkres, but also include the equivalent but more common form at the end of the section. We first require a definition:

Def Let A be a set with a strict partial order relation $\prec$. For $B \subset A$, an upper bound on B is a $c \in A$ such that $b \preceq c$ for all $b \in B$. We say $m \in A$ is a maximal element if $m \prec a$ holds for no $a \in A$.

Zorn's Lemma (Munkres Format) Let A be a set with a strict partial order relation. If every totally ordered subset of A has an upper bound in A, then A has a maximal element.

Proof By the Maximum Principle, A has a maximal totally ordered subset $B \subset A$. By assumption, B has an upper bound $c \in A$. We claim c is a maximal element of A. Indeed, if $c \prec d$ for some $d \in A$, then $B \cup \{d\}$ is totally ordered and strictly larger than B, contradicting the maximality of B. $\square$

Zorn's Lemma (Common Format) Let A be a set with a partial order relation. If every chain in A has an upper bound in A, then A has a maximal element.

- We've seen: $\boxed{\text{Axiom of Choice}} \Rightarrow \boxed{\text{Well-Ordering Theorem}} \Rightarrow \boxed{\text{Zorn's Lemma}}$