§12 Topological Spaces

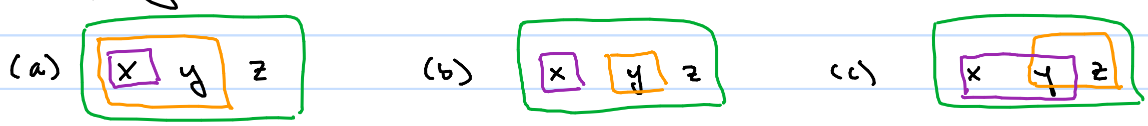
Def A <u>topology</u> on a set X is a collection $\mathcal{T}$ of subsets of X satisfying:

1. $\phi, X \in \mathcal{T}$.
2. For any subcollection $S \subset \mathcal{T}$, $\bigcup_{U \in S} U \in \mathcal{T}$.
3. For any $n \in \mathbb{N}$ and $U_1, \dots, U_n \in \mathcal{T}$, $U_1 \cap U_2 \cap \dots \cap U_n \in \mathcal{T}$.

A set X equipped with a topology $\mathcal{T}$ is called a <u>topological space</u>. We call the sets $U \in \mathcal{T}$ <u>open</u> sets (with respect to $\mathcal{T}$).

- So a topology on a set X is a choice of open sets such that ϕ and X are open, unions of open sets are open, and <u>finite</u> intersections of open sets are open.

Ex ① Let $X = \{x, y, z\}$. Which of the following collections are topologies? (assume they all include ϕ):



only (a)

② Let X be any set. Letting $\mathcal{T} := \mathcal{P}(X)$ be the power set of X (the collection of <u>all</u> subsets of X), defines a topology on X. In this case, <u>every</u> subset of X is open. This is called the <u>discrete</u> topology on X.
At the other extreme, we can let $\mathcal{T} := \{\phi, X\}$. That is, only ϕ and X are open. This is called the <u>trivial</u> (or <u>indiscrete</u>) topology.

③ For $X = \mathbb{R}$, let $\mathcal{T}$ be the collection of subsets $U \subset \mathbb{R}$ such that $\forall x \in U$ $\exists \varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subset U$.
This is a topology on $\mathbb{R}$:

1. $\phi \in \mathcal{T}$ because the condition holds vacuously, and $\mathbb{R} \in \mathcal{T}$ since $\forall x \in \mathbb{R}$, $(x-1, x+1) \subset \mathbb{R}$.
2. If $S \subset \mathcal{T}$ and $x \in \bigcup_{U \in S} U$, then $x \in U_0 \in S \subset \mathcal{T}$ for some U_0. Hence $\exists \varepsilon > 0$ so that
$$(x-\varepsilon, x+\varepsilon) \subset U_0 \subset \bigcup_{U \in S} U.$$
Thus $\bigcup_{U \in S} U \in \mathcal{T}$.
3. If $U_1, \dots, U_n \in \mathcal{T}$ and $x \in U_1 \cap \dots \cap U_n$, then

for each $j=1,\dots,n$ $\exists\, \varepsilon_j > 0$ so that

$$(x-\varepsilon_j, x+\varepsilon_j) \subset U_j$$

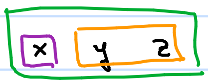
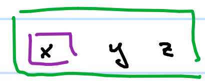
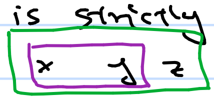
Let $\varepsilon := \min_{1\le j\le n} \varepsilon_j$. Then $(x-\varepsilon, x+\varepsilon) \subset U_j$ for each $j=1,\dots,n$. Hence

$$(x-\varepsilon, x+\varepsilon) \subset U_1 \cap U_2 \cap \cdots \cap U_n.$$

This is called the "standard" topology on $\mathbb{R}$. In your analysis class, $U \subset \mathbb{R}$ was open iff $U \in \mathcal{T}$. $\square$

Def Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on the same set X. We say $\mathcal{T}$ is <u>finer</u> than $\mathcal{T}'$ if $\mathcal{T} \supset \mathcal{T}'$. We also say $\mathcal{T}'$ is <u>coarser</u> than $\mathcal{T}$. If $\mathcal{T} \supsetneq \mathcal{T}'$, we say $\mathcal{T}$ is <u>strictly finer</u> than $\mathcal{T}'$, and $\mathcal{T}'$ is <u>strictly coarser</u> than $\mathcal{T}$.

- The discrete topology is finer than any other topology, while the trivial topology is coarser than any other topology.

Ex For $X = \{x, y, z\}$  is strictly finer than , but neither is comparable to  $\square$

Rem In analysis, one often sees topologies described as "stronger" (or "weaker") than one another. Typically, saying $\mathcal{T}$ is stronger than $\mathcal{T}'$ means $\mathcal{T}$ is finer than $\mathcal{T}'$, and weaker corresponds to coarser. The reasoning will be clearer once we discuss "convergence", which is how analysts typically describe topologies. $\square$

§ 13 Basis for a Topology

- Usually it is difficult (even nigh impossible) to explicitly describe a topology like we did in Ex 3 from last section. A more tractable (and practical) approach is to specify a subcollection $S \subset \mathcal{T}$ that "generates" $\mathcal{T}$ in some way. In this section we will discuss two ways to do this.

Def For a set X, a collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X if it satisfies:

1. For each $x \in X$, there is at least one $B \in \mathcal{B}$ with $x \in B$.
2. If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

In this case, the topology generated by $\mathcal{B}$ is the collection $\mathcal{T}$ of subsets $U \subset X$ such that for each $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subset U$.

- We need to check that the collection $\mathcal{T}$ defined above actually satisfies the definition of a topology, but we first consider some examples.

Ex 1. For a set X, let $\mathcal{B} := \{\{x\} : x \in X\}$. Then $\mathcal{B}$ is a basis:

1: $x \in \{x\}$ for all $x \in X$

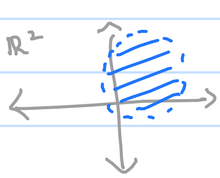
2: If $x \in \{y\} \cap \{z\}$ for some $x, y, z \in X$, then $x = y = z$ and so we have $x \in \{x\} \subset \{y\} \cap \{z\}$.

What is the topology generated by this basis? Discrete Top.

2. For a set X, let $\mathcal{B} := \{\{X\}\}$. Then $\mathcal{B}$ is a basis (Exercise: check this!)

What is the topology generated by this basis? Trivial Top.

3. Identifying $\mathbb{R}^2$ with the Euclidean plane, let $\mathcal{B}$ the collection of interiors of circles. Then $\mathcal{B}$ is a basis. $U \subset \mathbb{R}^2$ is open in the topology generated by $\mathcal{B}$ iff $\forall x \in U$ one can draw a circle around x that stays inside U.



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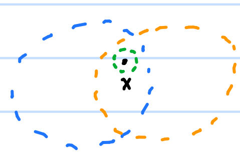
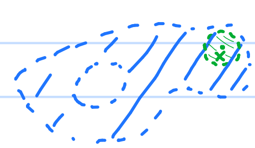
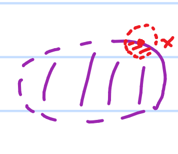


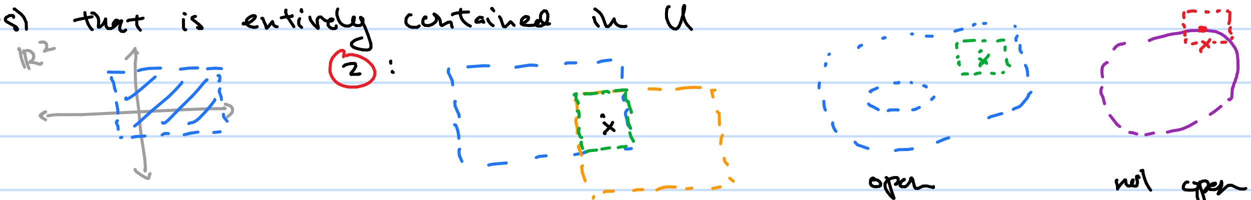


open



not open

4 In $\mathbb{R}^2$ again, let $\mathcal{B}'$ be the collection of interiors of rectangular regions with sides parallel to the x and y axes. Then $\mathcal{B}'$ is a basis and a set U is open in the topology generated by $\mathcal{B}'$ iff $\forall x \in U$ you can draw a rectangle around x (with sides parallel to the x and y axes) that is entirely contained in U



As the examples above suggest, $\mathcal{B}$ and $\mathcal{B}'$ generate the same topologies. We'll see a rigorous proof of this shortly. $\square$

Thm Let X be a set and $\mathcal{B}$ a basis for a topology on X. If $\mathcal{T}$ is the topology generated by $\mathcal{B}$, then it is actually a topology and $\mathcal{B} \subset \mathcal{T}$.

Proof Recall that $U \subset X$ is in $\mathcal{T}$ iff for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subset U$. Note that this holds for any $U \in \mathcal{B}$ by letting $B := U$. Thus $\mathcal{B} \subset \mathcal{T}$. Now, let us check $\mathcal{T}$ is a topology. We have $\phi \in \mathcal{T}$ because the condition holds vacuously. We also have $X \in \mathcal{T}$ because by definition of a basis, for all $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$ and this is of course contained in X.

Next we check $\mathcal{T}$ is closed under unions. Let $S \subset \mathcal{T}$ be a subcollection and let

$$U_1 := \bigcup_{U \in S} U$$

If $x \in U_1$, then $x \in U$ for some $U \in S \subset \mathcal{T}$. By definition of $\mathcal{T}$, there is some $B \in \mathcal{B}$ such that $x \in B \subset U$, and this is further contained in U_1: $x \in B \subset U_1$. Since $x \in U_1$ was arbitrary, we have $U_1 \in \mathcal{T}$.

Finally, we check $\mathcal{T}$ is closed under finite unions. Let $U_1, \ldots, U_n \in \mathcal{T}$ for some $n \in \mathbb{N}$. If $x \in U_1 \cap \cdots \cap U_n$, then $x \in U_j$ for each $j = 1, \ldots, n$. Invoking the definition of $\mathcal{T}$ again implies there are $B_1, \ldots, B_n \in \mathcal{B}$ such that $x \in B_j \subset U_j$ for each $j = 1, \ldots, n$. We claim there is a $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap \cdots \cap B_n$ and hence

$$x \in B \subset U_1 \cap \cdots \cap U_n,$$

which means $U_1 \cap \cdots \cap U_n \in \mathcal{T}$. The claim holds for $n = 2$, by the definition of a basis, and the general case follows from a simple induction proof. $\square$

- From the above theorem, if $\mathcal{B}$ is a basis and $\mathcal{T}$ is the topology it

generates then for any subcollection $\mathcal{C} \subset \mathcal{B} \subset \mathcal{T}$, we have $\bigcup_{U \in \mathcal{C}} U \in \mathcal{T}$.
It turns out every open set has this form.

Thm Let X be a set and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements in $\mathcal{B}$.

Proof Call this collection $\mathcal{T}'$. By the discussion preceding the lemma, we have $\mathcal{T}' \subset \mathcal{T}$. Conversely, given $U \in \mathcal{T}$, for all $x \in \mathcal{T}$ there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$ (by definition of $\mathcal{T}$). Thus

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} B_x \subset U,$$

which implies $U = \bigcup_{x \in U} B_x \in \mathcal{T}'$. Thus $\mathcal{T} \subset \mathcal{T}'$ and so $\mathcal{T} = \mathcal{T}'$. □

Cor Let X be a set and $\mathcal{B}$ a basis for a topology on X. Then the topology generated by $\mathcal{B}$ is coarser than any topology which contains $\mathcal{B}$.

Proof Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$, and let $\mathcal{T}'$ be a topology on X containing $\mathcal{B}$. Since $\mathcal{T}'$ is closed under taking unions, the previous theorem implies $\mathcal{T} \subset \mathcal{T}'$. Hence $\mathcal{T}$ is coarser than $\mathcal{T}'$. □

- Thus the topology generated by $\mathcal{B}$ is the smallest (coarsest) topology on X containing $\mathcal{B}$.

- We have now seen two ways to describe a topology in terms of a basis. It is possible to do the reverse as well.

Lemma Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open subsets of X such that for any open subset $U \subset X$ and any $x \in U$ there exists $C \in \mathcal{C}$ with $x \in C \subset U$. Then $\mathcal{C}$ is basis for the topology on X (i.e. $\mathcal{C}$ generates the existing topology on X).

Proof We first check $\mathcal{C}$ is a basis. Since X itself is open, the hypotheses imply for all $x \in X$ there exists $C \in \mathcal{C}$ with $x \in C \subset X$, which is 1 in the definition of a basis. Next, suppose $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open, so is $U := C_1 \cap C_2$. So by assumption $\exists\, C_3 \in \mathcal{C}$ with $x \in C_3 \subset U = C_1 \cap C_2$, which is 2 in the definition of basis. So $\mathcal{C}$ is a basis.
Now let $\mathcal{T}$ be the topology generated by $\mathcal{C}$, and let $\mathcal{T}'$ be the original topology on X. Since $\mathcal{C} \subset \mathcal{T}'$, we have $\mathcal{T} \subset \mathcal{T}'$ by the previous corollary.

Conversely, if $U \in \mathcal{T}'$ then by assumption for all $x \in U$ $\exists\, C_x \in \mathcal{C}$ with $x \in C_x \subset U$. Thus $U = \bigcup_{x \in U} C_x \in \mathcal{T}$. Hence $\mathcal{T}' \subset \mathcal{T}$ and so $\mathcal{T} = \mathcal{T}'$. □

- Now that we can understand topologies in terms of bases (and vice versa) it is possible to compare topologies using only bases:

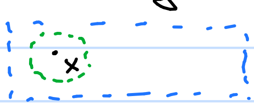
Lemma Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$ $(\mathcal{T}' \supset \mathcal{T})$.

(ii) For each $x \in X$ and each $B \in \mathcal{B}$ containing x, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof $(i \Rightarrow ii)$: Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Then $\mathcal{B} \subset \mathcal{T} \subset \mathcal{T}'$, so every $B \in \mathcal{B}$ is open with respect to $\mathcal{T}'$. Fix $B \in \mathcal{B}$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, for all $x \in B$ there exists $B' \in \mathcal{B}'$ with $x \in B' \subset B$.
$(ii \Rightarrow i)$: This means $B \in \mathcal{T}'$ for all $B \in \mathcal{B}$. Thus $\mathcal{B} \subset \mathcal{T}'$, and so the corollary implies $\mathcal{T}$ (the topology generated by $\mathcal{B}$) is coarser than $\mathcal{T}'$. Equivalently, $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Ex In $\mathbb{R}^2$, let $\mathcal{B}$ be the collection of all interiors of circles in the plane, and let $\mathcal{B}'$ be the collection of all interiors of rectangles with sides parallel to the x and y axes. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies generated by $\mathcal{B}$ and $\mathcal{B}'$, respectively. Using the lemma, we have



$\Rightarrow$ $\mathcal{T}$ is finer than $\mathcal{T}'$

$\Rightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.

Hence $\mathcal{T} = \mathcal{T}'$; that is, $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology. □

- We now consider three topologies on $\mathbb{R}$

Def The standard topology on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}$ consisting of open intervals:

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

We will always consider $\mathbb{R}$ with this topology unless we specifically say otherwise.

The lower limit topology on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}'$ consisting of half-open intervals of the form

$$[a, b) := \{x \in \mathbb{R} \mid a \le x < b\}$$

We write $\mathbb{R}_\ell$ to denote $\mathbb{R}$ equipped with this topology.

Let $K := \{ \frac{1}{n} \mid n \in \mathbb{N} \}$. The K-topology on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}''$ consisting of open intervals (a,b) and sets of the form $(a,b) \setminus K$. We write $\mathbb{R}_K$ to denote $\mathbb{R}$ equipped with this topology.

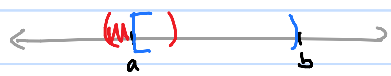
• Exercise Check $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$ are all bases.

Lemma The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

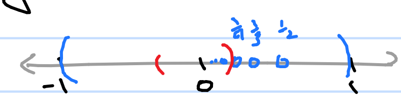
Proof We'll use the previous lemma. If $x \in (a,b)$, then

$$x \in [x,b) \subset (a,b)$$

Since $[x,b)$ belongs to the basis generating the lower limit topology, we see that this is finer than the standard topology. Moreover, it is strictly finer: $[a,b)$ is in the lower limit topology by definition, but not in the standard topology because there is no open interval containing a and contained in $[a,b)$:



Similarly for $x \in (a,b)$ we have $x \in (a,b) \subset (a,b)$. Since (a,b) is in the K-topology, we see that the K-topology is finer than the standard topology. To see that it is strictly finer, note that $(-1,1)\setminus K$ is in the K topology, but not in the standard topology. Indeed $0 \in (-1,1)\setminus K$ but any interval containing 0 will intersect K and thus fail to be contained in $(-1,1)\setminus K$:



Finally, $[1,2)$ is in the lower limit topology but not in the K-topology, while $(-1,1)\setminus K$ is in the K-topology but not the lower limit topology (Exercise check this). Thus the topologies are not comparable. $\square$

• We conclude with another method for generating topologies.

Def Let X be a set. A subbasis for a topology on X is a collection S of subsets of X whose union is all of X. The topology generated by the subbasis S is the collection $\mathcal{T}$ of all unions of finite intersections of elements of S.

Prop Let S be a subbasis for a topology on X. If $\mathcal{T}$ is the collection of all unions of finite intersections of elements of S, then $\mathcal{T}$ is a topology.

Proof Define $\mathcal{B}$ to the collection of all finite intersections of elements of S.

We claim $\mathcal{B}$ is a basis. Indeed, $\bigcup_{U \in S} U = X$ so for all $x \in X$ there is some $U \in S \subset \mathcal{B}$ with $x \in U$. Also, if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then B_1 and B_2 are finite intersections of elements of S. Consequently, so is $B_1 \cap B_2$. That is, $B_3 := B_1 \cap B_2 \in \mathcal{B}$ and $x \in B_3 \subset B_1 \cap B_2$. Hence $\mathcal{B}$ is a basis. Since $\mathcal{T}$ is the collection of unions of elements of $\mathcal{B}$, it is the topology generated by $\mathcal{B}$ by the second theorem in this section. $\square$

- Thus the topology generated by a subbasis S is the same as the topology generated by the basis $\mathcal{B}$ consisting of finite intersections of elements in S.

§14 The Order Topology

- Throughout this section, X will be a set with order relation $<$. In addition to the open interval notation (a,b) we introduced in §3, we will also write for $a,b \in X$ with $a<b$:

$$[a,b] := \{x \in X \mid a \le x \le b\}$$
$$(a,b] := \{x \in X \mid a < x \le b\}$$
$$[a,b) := \{x \in X \mid a \le x < b\}$$

There is a natural topology on X which respects the order relation in the sense that the open intervals (a,b) will actually be open in this topology.

Def Let X be a set with an order relation $<$. Assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

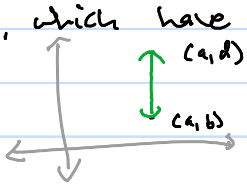
1. Open intervals (a,b) in X.
2. Half-open intervals $[a_0, b)$, where a_0 is the smallest element (if any) of X.
3. Half-open intervals $(a, b_0]$, where b_0 is the largest element (if any) of X.

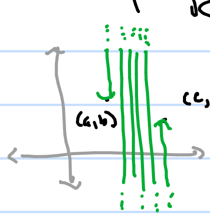
The collection $\mathcal{B}$ is a basis for a topology on X, which we call the order topology.

- We must verify that $\mathcal{B}$ is a basis. First, in X has a smallest (resp. largest) element, it is contained in intervals of type 2 (resp. 3). All other elements are necessarily contained in an open interval (since otherwise they are either the smallest or largest element). So the first condition of being a basis holds, and the second condition follows from the fact that a finite intersection of intervals of the above types is either an interval of the above types or empty (Exercise check the various cases for pairs of intervals).

Ex 1. When $\mathbb{R}$ has its usual order, the order topology is generated by the basis consisting of all open intervals (since $\mathbb{R}$ has no smallest or largest elements). Thus the order topology is the same as the standard topology on $\mathbb{R}$.

2. Let $\mathbb{R}\times\mathbb{R}$ have the dictionary order. There are no largest or smallest elements, so the order topology is generated by open intervals, which have two forms:





Is this comparable to the topology generated by the interiors of circles? Yes, it is finer.

(3) When $\mathbb{N}$ has its usual order, the order topology equals the discrete topology: $\{n\} = (n-1, n+1)$ is open for $n \geq 2$, and so is $\{1\} = [1,2)$. Since any subset of $\mathbb{N}$ is a union of singletons, all subsets of $\mathbb{N}$ are open.

(4) When $\{1,2\} \times \mathbb{N}$ has the dictionary order, the order topology is not the discrete topology: $\{(2,1)\}$ is not open. $\square$

Def Let X be a set with an order relation $<$. For $a \in X$, the rays determined by a are the subsets

$$(a,+\infty) := \{x \in X \mid a < x\}$$
$$(-\infty, a) := \{x \in X \mid x < a\}$$

open rays

$$[a, +\infty) := \{x \in X \mid a \leq x\}$$
$$(-\infty, a] := \{x \in X \mid x \leq a\}.$$

closed rays

Prop Let X be a set with order relation $<$ and at least two elements. Then the topology generated by the subbasis consisting of open rays is the order topology.

Proof We first check the collection S of open rays is a subbasis. This just means the union of all open rays is X. Let $a, b \in X$ with $a < b$. Then

$$(a,+\infty) \cup (-\infty, b) = X,$$

and so the collection of open rays in X is a subbasis. Observe

$$(a,b) = (a,+\infty) \cap (-\infty, b)$$
$$(a, b_0] = (a, +\infty)$$
$$[a_0, b) = (-\infty, b)$$

where a_0 and b_0 are the smallest and largest elements of X, respectively, if they exist. Otherwise

$$(a,+\infty) = \bigcup_{b > a} (a,b)$$
$$(-\infty, a) = \bigcup_{c < a} (c, a)$$

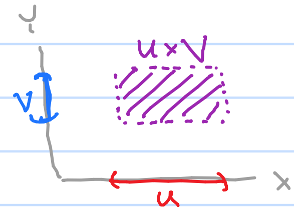
This shows the topology generated by S contains the basis for the order topology, and the order topology contains S. By Exercise 2 on Homework 3, the two topologies must be equal. $\square$

§15 The Product Topology (part 1)

- Given two topological spaces X and Y, we will discuss how to define a topology on their cartesian product

$$X \times Y = \{ (x,y) \mid x \in X, y \in Y \}$$

Later (in §19), we'll discuss how to do this for cartesian products of arbitrary size (instead of just size 2).



- Given open sets $U \subset X$ and $V \subset Y$, it would be natural for $U \times V$ to also be open. If we can show the collection

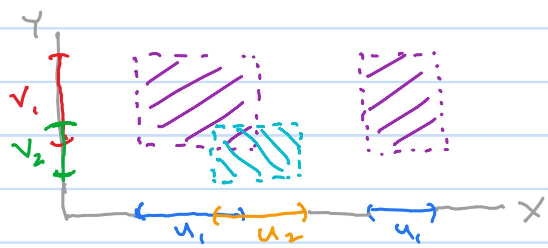
$$\mathcal{B} := \{ U \times V \mid U \subset X \text{ open}, V \subset Y \text{ open} \}$$

is a basis, then $U \times V$ will be open in the topology it generates. First note $X \times Y \in \mathcal{B}$, so every element of $X \times Y$ is contained in a set from $\mathcal{B}$. Next, suppose $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ are open. Observe

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (\underbrace{U_1 \cap U_2}_{\text{open}}) \times (\underbrace{V_1 \cap V_2}_{\text{open}})$$

so this intersection belongs to $\mathcal{B}$. Hence $\mathcal{B}$ is a basis.

Def Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology $\mathcal{T}$ generated by the basis consisting of subsets of the form $U \times V$ for $U \subset X$ open and $V \subset Y$ open.



$U_1 \times V_1 \in \mathcal{B}$ but $(U_1 \times V_1) \cup (U_2 \times V_2) \in \mathcal{T} \setminus \mathcal{B}$.

$U_2 \times V_2 \in \mathcal{B}$

- We've seen it is useful to describe topologies via bases or subbases. So if the topologies on X and Y are defined this way, it will be convenient to describe the product topology on $X \times Y$ directly in terms of their bases:

Thm Let $\mathcal{B}$ be a basis for the topology on X, and let $\mathcal{C}$ be a basis for the topology on Y. Then the collection

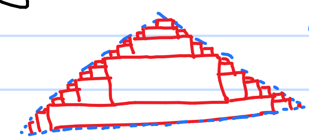
$$\mathcal{D} := \{ B \times C \mid B \in \mathcal{B}, C \in \mathcal{C} \}$$

is a basis for the topology on $X \times Y$.

Proof We will invoke the first lemma in §13: given an open set $W \subset X \times Y$ and $(x,y) \in W$, we must find $B \times C \in \mathcal{D}$ such that $(x,y) \in B \times C \subset W$. By definition of the product topology, there are open sets $U \subset X$, $V \subset Y$ such that $(x,y) \in U \times V \subset W$. Since $\mathcal{B}$ is a basis for the topology on X, there exists $B \in \mathcal{B}$ with $x \in B \subset U$. Similarly, there exists $C \in \mathcal{C}$ with $y \in C \subset V$. Consequently, $B \times C \in \mathcal{D}$ and $(x,y) \in B \times C \subset U \times V \subset W$. $\square$

Ex Let $\mathbb{R}$ have the standard topology. The product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is called the standard topology on $\mathbb{R}^2$. Since open intervals $(a,b) \subset \mathbb{R}$ generate the standard topology on $\mathbb{R}$, the theorem implies products of intervals $(a,b) \times (c,d)$ generate the standard topology on $\mathbb{R}^2$. But $(a,b) \times (c,d)$ is simply the interior of a rectangle with sides parallel to the x and y axes, so this topology on $\mathbb{R}^2$ is the same as the one generated by interiors of circles.

Note that most sets that "look" open in $\mathbb{R}^2$ actually are because you can express them as unions of $(a,b) \times (c,d)$.



open triangle is open

$\square$

Def The functions

$$\pi_1 : X \times Y \to X \qquad\qquad \pi_2 : X \times Y \to Y$$
$$(x,y) \mapsto x \qquad\qquad (x,y) \mapsto y$$

are called (coordinate) projections

- Note that π_1 and π_2 are both surjective (assuming $X \neq \emptyset \neq Y$), and for subsets $S \subset X$ and $T \subset Y$ we have

$$\pi_1^{-1}(S) = S \times Y \quad \text{and} \quad \pi_2^{-1}(T) = X \times T.$$

In particular, for $U \subset X$ and $V \subset Y$ open we have

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

This hints towards a subbasis for the product topology:

Thm Let X and Y be topological spaces. The collection

$$\mathcal{S} := \{\pi_1^{-1}(U) \mid U \subset X \text{ open}\} \cup \{\pi_2^{-1}(V) \mid V \subset Y \text{ open}\}$$

is a subbasis for the product topology on $X \times Y$.

Proof Note that $X \times Y = \pi_1^{-1}(X) \in \mathcal{S}$, and so $\mathcal{S}$ is subbasis. Now, let $\mathcal{T}$ be the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$, $\mathcal{S} \subset \mathcal{T}$. Since $\mathcal{T}$ is closed under finite intersections and arbitrary unions, it follows that $\mathcal{T}' \subset \mathcal{T}$. Conversely, for open sets $U \subset X$ and $V \subset Y$, we saw in the discussion preceding the theorem that $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, hence $U \times V \in \mathcal{T}'$. Since such sets generate $\mathcal{T}$ we have $\mathcal{T} = \mathcal{T}'$. $\square$

§16 The Subspace Topology

Prop Let X be a topological space with topology $\mathcal{T}$. For a subset $Y \subset X$ the collection
$$\mathcal{T}_Y := \{ Y \cap U \mid U \in \mathcal{T} \}$$
is a topology on Y.

Proof Since $\phi, X \in \mathcal{T}$, we have $\phi = Y \cap \phi \in \mathcal{T}_Y$ and $Y = Y \cap X \in \mathcal{T}_Y$. If $S \subset \mathcal{T}_Y$ is a subcollection, then every element in S is of the form $Y \cap U$ for $U \in \mathcal{T}$ and hence
$$\bigcup_{Y \cap U \in S} (Y \cap U) = Y \cap \Big(\overbrace{\bigcup_{Y \cap U \in S} U}^{\in \mathcal{T}} \Big) \in \mathcal{T}_Y.$$
Finally, for $Y \cap U_1, \dots, Y \cap U_n \in \mathcal{T}_Y$ we have
$$(Y \cap U_1) \cap \cdots \cap (Y \cap U_n) = Y \cap \overbrace{(U_1 \cap \cdots \cap U_n)}^{\in \mathcal{T}} \in \mathcal{T}_Y.$$
Thus $\mathcal{T}_Y$ is a topology on Y. $\square$

Def Let X be a topological space with topology $\mathcal{T}$. For a subset $Y \subset X$, the topology
$$\mathcal{T}_Y = \{ Y \cap U \mid U \in \mathcal{T} \}$$
on Y is called the subspace topology. When equipped with this topology, we call Y a (topological) subspace.

- Taking intersections with Y also turns a basis for the topology on X into a basis for the subspace topology on Y:

Lemma If $\mathcal{B}$ is a basis for the topology on X and $Y \subset X$, then
$$\mathcal{B}_Y := \{ Y \cap B \mid B \in \mathcal{B} \}$$
is a basis for the subspace topology on Y.

Proof Let $U \subset X$ be open and let $y \in Y \cap U$. Then there exists $B \in \mathcal{B}$ with $y \in B \subset U$. So $Y \cap B \in \mathcal{B}_Y$ and $y \in Y \cap B \subset Y \cap U$. Thus $\mathcal{B}_Y$ is a basis for the subspace topology on Y by the first lemma in §13. $\square$

- Since $Y \in \mathcal{T}_Y$, Y is always open in the subspace topology, even if it wasn't open in $\mathcal{T}$. More generally, we can have $U \in \mathcal{T}_Y$ but $U \notin \mathcal{T}$. Thus we may say $U \subset Y$ is open in Y (or relative to Y) if $U \in \mathcal{T}_Y$, and say U is open in X if $U \in \mathcal{T}$.

Ex Let $\mathbb{R}$ have the standard topology. Then $[0,1)$ is not open in $\mathbb{R}$, but it is open in $[0,+\infty)$ and is open in $[0,1]$. $\square$

- An exception to the above occurs when Y is actually open in X:

Lemma Let Y be a subspace of a topological space X. If Y is open in X, then $U \subset Y$ is open in Y if and only if it is open in X.

Proof Assume Y is open in X. If $U \subset Y$ is open in X, then $U = Y \cap U$ is open in Y (in fact, this holds regardless of Y being open). Conversely, if U is open in Y, then $U = Y \cap V$ for some V open in X. But then U is open in X as the finite intersection of open sets Y, V. $\square$

- The subspace topology interacts "nicely" with the product topology, as the following theorem shows.

Thm Let X and Y be topological spaces with subspaces $A \subset X$ and $B \subset Y$. Equip $X \times Y$ with the product topology. The subspace topology on $A \times B \subset X \times Y$ is the same as the product topology on $A \times B$ when A and B are each given the subspace topology.

Proof We'll show these topologies have a common basis. Recall $U \times V$ for open sets $U \subset X$ and $V \subset Y$ is a generic basis element for the product topology on $X \times Y$. Thus

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$$

is a generic basis element for the subspace topology on $A \times B$. But $A \cap U$ and $B \cap V$ are generic open sets in the respective subspace topologies on A and B. Thus $(A \cap U) \times (B \cap V)$ is also generic basis element for the product topology on $A \times B$. Since a basis only generates one topology, the two topologies must be equal. $\square$

- The above theorem tells us that whether we view $A \times B$ as a subspace of a product or a product of subspaces, we'll get the same topology.

- Let X be a set with order relation $<$, and equip X with the order topology. Given $Y \subset X$, the relation $<$ restricts to an order relation on Y. Thus we can either give Y the subspace topology or its own order topology. Unfortunately, these need not always be the same.

Ex 1 Let $\mathbb{R}$ have the standard topology, which we recall is the same as its order topology when given the usual order. For $Y = [0,1]$, the subspace topology and order topologies agree: they have a common basis consisting of sets of the form

$$Y \cap (a,b) = \begin{cases} (a,b) & \text{if } a,b \in Y \\ (a,1] & \text{if only } a \in Y \\ [0,b) & \text{if only } b \in Y \\ \phi & \text{if neither are in } Y \end{cases}$$

2 For $Y := [0,1) \cup \{2\} \subset \mathbb{R}$, the subspace and order topologies are distinct. Indeed, $\{2\} = Y \cap (\frac{3}{2}, \frac{5}{2})$ is open in the subspace topology. However, any basis element in the order topology on Y that contains 2 is of the form $Y \cap (a,2]$ for some $a \in Y$ with $a < 2$. But $a \in Y$ and $a < 2$ together imply $a \in [0,1)$. Consequently, $Y \cap [a,2) \not\subset \{2\}$. Thus $\{2\}$ is not open because it doesn't contain any basis subsets. $\square$

- A subtle but important distinction between the above two examples was that the subspace in 1 did not have any gaps while the subspace in 2 did.

Def Let X be an ordered set. We say $Y \subset X$ is convex if whenever $a,b \in Y$ with $a < b$, one has $(a,b) \subset Y$.

Thm Let X be an ordered set equipped with the order topology. Let $Y \subset X$ be convex. Then the order and subspace topologies on Y are equal.

Proof Recall that the open rays $(-\infty, a)$ and $(a, +\infty)$ form a subbasis for the order topology. It follows that $Y \cap (-\infty, a)$ and $Y \cap (a, +\infty)$ form a subbasis for the subspace topology on Y (Exercise confirm this).

If $a \in Y$, then

$$Y \cap (-\infty, a) = \{y \in Y \mid y < a\}$$
$$Y \cap (a, +\infty) = \{y \in Y \mid a < y\}$$

which are open in the order topology on Y. If $a \notin Y$, then the convexity of Y implies either $a < y$ for all $y \in Y$ or $y < a$ for all $y \in Y$. Indeed, otherwise there are $y_1, y_2 \in Y$ with $y_1 < a < y_2$, and so $a \in (y_1, y_2) \subset Y$, a contradiction. Thus we have

$$Y \cap (-\infty, a) = \begin{cases} \phi & \text{if } a < y \ \forall y \in Y \\ Y & \text{if } y < a \ \forall y \in Y \end{cases} \qquad Y \cap (a, +\infty) = \begin{cases} Y & \text{if } a < y \ \forall y \in Y \\ \phi & \text{if } y < a \ \forall y \in Y. \end{cases}$$

Thus the sets $Y \cap (-\infty, a)$ and $Y \cap (a, +\infty)$ are always open in the order topology. Since these are a subbasis for the subspace topology, we see it is coarser than the order topology on Y.

Conversely, the open rays in Y satisfy for $a \in Y$

$$\{y \in Y \mid y < a\} = Y \cap (-\infty, a) \qquad \{y \in Y \mid a < y\} = Y \cap (a, +\infty)$$

and hence are open in the subspace topology. Since the open rays in Y form a subbasis for the order topology in Y, we see that it is coarser than the subspace topology. Hence the two topologies are equal. $\square$

Rem We take the convention that if X has the order topology, then a subspace $Y \subset X$ will have the subspace topology unless we specifically state otherwise. (If Y is convex, the above theorem says there is no potential ambiguity).

§17 Closed Sets and Limit Points

Def Let X be a topological space. We say a subset $A \subset X$ is closed if its complement $X \setminus A$ is open.

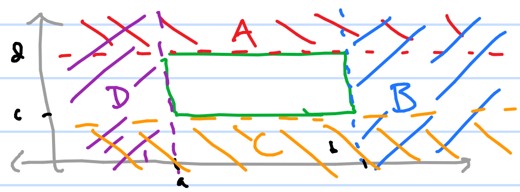
Ex ① A closed interval $[a,b] \subset \mathbb{R}$ is closed since its complement $(-\infty, a) \cup (b, +\infty)$ is open (as the union of open rays).
However, the half-open interval $[a,b) \subset \mathbb{R}$ is not closed (nor is it open) because its complement $(-\infty, a) \cup [b, +\infty)$ is not open: $\forall \varepsilon > 0$ $(b-\varepsilon, b+\varepsilon) \not\subset (-\infty, a) \cup [b, +\infty)$.
$\mathbb{R}$ itself is closed since its complement $\emptyset$ is open.
Thus sets can be closed, open, neither, or both.

② The closed rectangle $[a,b] \times [c,d] \subset \mathbb{R}^2$ is closed since its complement is the union of four open sets:

$$A \cup B \cup C \cup D$$

$A = \mathbb{R} \times (d, +\infty)$ $\quad C = \mathbb{R} \times (-\infty, c)$

$B = (b, +\infty) \times \mathbb{R}$ $\quad D = (-\infty, a) \times \mathbb{R}$



③ If X has the discrete topology, then every set is closed since all sets (including complements) are open.
If X has the trivial topology, the only closed sets are X and $\emptyset$, since their complements are the only open sets. □

• There is a symmetry to open and closed sets, since they are complements of one another. This symmetry yields the following properties for the collection of closed sets, which are analogous to the defining properties for the collection of open sets (i.e. the topology)

Thm Let X be a topological space.
① $\emptyset$ and X are closed
② If S is a collection of closed sets, then $\bigcap_{A \in S} A$ is closed.
③ If $A_1, \ldots, A_n \subset X$ are closed, then $A_1 \cup \cdots \cup A_n$ is closed.

Proof ①: Since X and $\emptyset$ are open, $\emptyset = X \setminus X$ and $X = X \setminus \emptyset$ are closed

②: Observe that

$$X \setminus \bigcap_{A \in S} A = \bigcup_{A \in S} (X \setminus A)$$

Since each $A \in S$ is closed, $X \setminus A$ is open and therefore so is their union. Thus the above implies $\bigcap_{A \in S} A$ is closed.

③: Observe

$$X \setminus (A_1 \cup \cdots \cup A_n) = (X \setminus A_1) \cap \cdots \cap (X \setminus A_n)$$

which is open as a finite intersection of open sets $X \setminus A_1, \ldots, X \setminus A_n$. Thus $A_1 \cup \cdots \cup A_n$ is closed. $\square$

- Let $Y \subset X$ be a subspace of a topological space. We say $A \subset Y$ is closed in Y if A is a closed set in the subspace topology on Y. That is, if $Y \setminus A$ is open in Y. We have the following characterization for being closed in Y, which analogous to the characterization of being open in Y.

$\boxed{\text{Thm}}$ Let Y be a subspace of a topological space X. Then a subset $A \subset Y$ is closed in Y if and only if $A = Y \cap B$ for some B that is closed in X.

Proof ($\Rightarrow$): Assume A is closed in Y. Then $Y \setminus A$ is open in Y and therefore $Y \setminus A = Y \cap U$ for some open subset $U \subset X$. Consider $B := X \setminus U$, which is closed. Then

$$Y \cap B = Y \setminus U = Y \setminus (Y \cap U) = Y \setminus (Y \setminus A) = A.$$

($\Leftarrow$): Suppose $A = Y \cap B$ for some subset B that is closed in X. Then

$$Y \setminus A = Y \setminus (Y \cap B) = Y \setminus B = Y \cap (X \setminus B)$$

Since $X \setminus B$ is open, $Y \setminus A$ is open in Y. Thus A is closed in Y. $\square$

- Recall that it was possible for a set $U \subset Y$ to be open in Y but not open X. The same thing can happen for closed sets (e.g. $(0, \frac{1}{2}]$ is closed in $(0,1)$ but not in $\mathbb{R}$). This cannot happen if Y itself is closed:

$\boxed{\text{Thm}}$ Let Y be a subspace of X. If Y is closed in X, then A is closed in Y if and only if A is closed in X.

Proof Assume A is closed in Y. The previous theorem implies there is a closed set $B \subset X$ such that $A = Y \cap B$. But then A is closed as the intersection of two closed sets. Conversely, if A is closed in X, then $A = Y \cap A$ is closed in Y by the previous theorem. $\square$

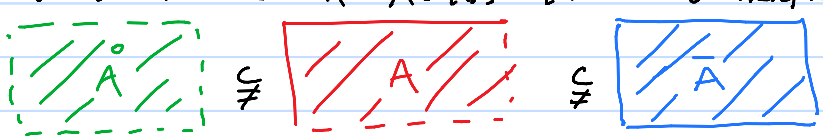
Closure and Interior of a Set

Def Let X be a topological space and $A \subset X$ a subset. The <u>interior</u> of A is the union of all open sets contained in A, and is denoted $\operatorname{Int} A$ or A°.
The <u>closure</u> of A is the intersection of all closed sets containing A and is denoted $\operatorname{Cl} A$ or $\bar{A}$.

- Note that one always has
$$A^\circ \subset A \subset \bar{A}$$
Also $A = A^\circ$ iff A is open and $A = \bar{A}$ iff A is closed (Exercise)

Ex ① Let $A := [a,b) \subset \mathbb{R}$. Then $A^\circ = (a,b)$ and $\bar{A} = [a,b]$.
Indeed, $(a,b) \subset A$ and is open so $(a,b) \subset A^\circ$. Also $a \notin A^\circ$ since for any open set $U \ni a$, we have $(a-\varepsilon, a+\varepsilon) \subset U$ for some $\varepsilon > 0$.
But then $U \not\subset A$. Thus we cannot have $a \in A^\circ$. So $A^\circ \subset A \setminus \{a\} = (a,b)$ which implies $A^\circ = (a,b)$.
For $\bar{A}$, we have $\bar{A} \subset [a,b]$ since $[a,b]$ is closed and contains A.
Also $b \in \bar{A}$ because if $B \supset A$ is closed then $b \in B$ since otherwise $(b-\varepsilon, b+\varepsilon) \in \mathbb{R} \setminus B$ for some $\varepsilon > 0$. But then $A \cap (\mathbb{R} \setminus B) \neq \emptyset$ which contradicts $B \supset A$. So $\bar{A} \supset A \cup \{b\} = [a,b]$ and therefore $\bar{A} = [a,b]$.

② In $\mathbb{R}^2$:



□

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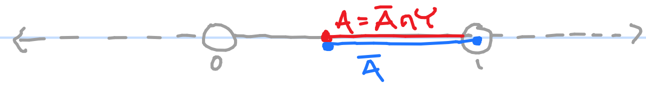
- If A is contained in some subspace $Y \subset X$, the notation $\bar{A}$ may seem ambiguous: do we mean the closure of A in X or the closure of A in Y? We will always take $\bar{A}$ to mean the closure in X since the closure in Y is simply $\bar{A} \cap Y$, as the following theorem shows:

Thm Let Y be a subspace of a topological space X. For $A \subset Y$, if $\bar{A}$ is the closure of A in X, then $\bar{A} \cap Y$ is the closure of A in Y.

<u>Proof</u> Let B denote the closure of A in Y. We'll show $B \subset \bar{A} \cap Y \subset B$.
$\bar{A} \cap Y$ is closed in Y since $\bar{A}$ is closed in X, and $\bar{A} \cap Y$ contains A.
Thus $B \subset \bar{A} \cap Y$ because B is the intersection of all such sets. Next, B is closed in Y so $B = C \cap Y$ for some closed set $C \subset X$. So $A \subset B \subset C$, and thus $\bar{A} \subset C$ since $\bar{A}$ is the intersection of closed sets containing A.
Consequently $\bar{A} \cap Y \subset C \cap Y = B$. Thus $B = \bar{A} \cap Y$. □

Ex In $\mathbb{R}$ we saw $\overline{[a,b)} = [a,b]$. For $Y = (0,1)$, $A = [\frac{1}{2}, 1)$ we have $\bar{A} = [\frac{1}{2}, 1]$ but $\bar{A} \cap Y = [\frac{1}{2}, 1) = A$. $\square$



- As we saw in Ex ① on the previous page, computing $\bar{A}$ directly can be difficult, even if A has a nice description (like an interval). The underlying issue is that there may be lots of closed sets, making it difficult to determine the intersection of even a subcollection of them. The following offers a more direct way to find $\bar{A}$.

Def In a topological space X, a neighborhood of $x \in X$ is an open set containing x.

Thm Let A be a subset of a topological space X.
① Then $x \in \bar{A}$ if and only if every neighborhood of x intersects A.
② Suppose $\mathcal{B}$ is a basis generating the topology on X. Then $x \in \bar{A}$ if and only if every $B \in \mathcal{B}$ with $x \in B$ satisfies $B \cap A \neq \emptyset$.

Proof
①: We prove each part via contrapositive. Suppose $x \notin \bar{A}$. Then by definition of $\bar{A}$, there exists a closed set $A \subset C \subset X$ with $x \notin C$. Then $x \in X \setminus C$, and so $X \setminus C$ is a neighborhood of x. Since $A \subset C$, we have $X \setminus C \subset X \setminus A$ and therefore $(X \setminus C) \cap A = \emptyset$. Conversely, suppose there is a neighborhood U of x such that $U \cap A = \emptyset$. Then $x \notin X \setminus U$ which is a closed set containing A. Thus $x \notin \bar{A}$.
②: Suppose $x \in \bar{A}$. Let $B \in \mathcal{B}$ be such that $x \in B$. Then B is a neighborhood of x and thus $B \cap A \neq \emptyset$ by the previous part. Conversely, suppose $x \notin \bar{A}$. Then by the previous part there exists a neighborhood U of x such that $U \cap A = \emptyset$. Since $\mathcal{B}$ is a basis, there exists $B \in \mathcal{B}$ satisfying $x \in B \subset U$. Consequently $B \cap A \subset U \cap A = \emptyset$. $\square$

Ex ① Let $A := (a,b) \subset \mathbb{R}$ be an open interval. Note that $\forall \varepsilon > 0$, $(a-\varepsilon, a+\varepsilon) \cap A \neq \emptyset$ and $(b-\varepsilon, b+\varepsilon) \cap A \neq \emptyset$. Thus $a, b \in \bar{A}$, and so $[a,b] \subset \bar{A}$. Since $[a,b]$ is a closed set containing A, we see $\bar{A} = [a,b]$.
One can similarly show $\overline{(a,b]} = [a,b]$.

② Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Then $0 \in \bar{A}$ since $\forall \varepsilon > 0$ we have

$$\tfrac{1}{n} \in (-\varepsilon, \varepsilon) \cap A$$

for $n \geq \lceil \frac{1}{\varepsilon} \rceil$. Thus $A \cup \{0\} \subset \bar{A}$. This is actually an equality since for any $x \notin A \cup \{0\}$, there exists $\varepsilon > 0$ so that $|x - \frac{1}{n}| \geq \varepsilon$ (check this) Hence $(x-\varepsilon, x+\varepsilon) \cap A = \emptyset$. $\square$

Limit Points

Def Let X be a topological space and $A \subset X$ a subset. We say $x \in X$ is a limit point (or cluster point) of A if for every neighborhood U of x contains an element of $A \setminus \{x\}$.

- By the previous theorem, x is a limit point of A iff $x \in \overline{A \setminus \{x\}}$. Also note that x does not need to be in A for it to be a limit point of A. In fact, some elements of A will not be limit points of A.

EX Let $A = [0,1) \cup \{2\} \subset \mathbb{R}$, where $\mathbb{R}$ has the standard topology. Then 1 is a limit point of A since $(1-\varepsilon, 1+\varepsilon) \cap (A \setminus \{1\}) = (1-\varepsilon, 1+\varepsilon) \cap A \neq \phi$ for all $\varepsilon > 0$. On the other hand, 2 is not a limit point of A (even though $2 \in A$), since $(2 - \frac{1}{2}, 2 + \frac{1}{2}) \cap (A \setminus \{2\}) = \phi$.
The set of all limit points of A is $[0,1]$. (Exercise verify!) $\square$

Thm Let X be a topological space with subset A. Let A' be the set of all limit points of A. Then

$$\bar{A} = A \cup A'.$$

Proof By the discussion following the definition of a limit point, for all $x \in A'$ we have $x \in \overline{A \setminus \{x\}} \subset \bar{A}$ (where the inclusion follows from Homework 4). Thus $A' \subset \bar{A}$ and of course $A \subset \bar{A}$. Hence $A \cup A' \subset \bar{A}$.
Conversely, let $x \in \bar{A}$ and we will show $x \in A \cup A'$. If $x \in A$ we are done, so suppose $x \notin A$. By the previous theorem every neighborhood U of x intersects A. So $\phi \neq U \cap A = U \cap (A \setminus \{x\})$ since $x \notin A$. Thus $x \in A'$, and so $\bar{A} \subset A \cup A'$. $\square$

Cor A subset of a topological space is closed if and only if it contains all of its limit points.
Proof If A is closed, then by the previous theorem $A = \bar{A} = A \cup A'$ where A' are the limit points of A. Thus $A' \subset A$. Conversely, if $A' \subset A$ then $A = A \cup A' = \bar{A}$, and so A is closed. $\square$

- In your analysis class you probably showed $A \subset \mathbb{R}$ is closed if and only if whenever $(a_n)_{n \in \mathbb{N}} \subset A$ is a convergent sequence one has $\lim_{n \to \infty} a_n \in A$. We can generalize this to arbitrary topological space, but it requires a generalization of sequences known as "nets."

Nets and Convergence

- In this section we discuss a generalization of sequences called "nets" which you secretly encountered back in calculus. Note that a sequence $(x_n)_{n\in\mathbb{N}}$ in a set X is really a function $x: \mathbb{N} \to X$ where we write $x_n := x(n)$. Nets generalize sequences by replacing $\mathbb{N}$ and its usual order relation with more generic sets equipped with a relation.

Def A directed set is a set I equipped with a relation $\leq$ satisfying:
1. $i \leq i$ for all $i \in I$ (Reflexivity)
2. If $i \leq j$ and $j \leq k$, then $i \leq k$. (Transitivity)
3. For any $i, j \in I$ there exists $k \in I$ with $i, j \leq k$ (Upper Bound Property)

- The upper bound property is a loosening of comparability: for $i, j \in I$ it may be that neither $i \leq j$ nor $j \leq i$ holds, but you can at least find $k \in I$ that you can compare i and j to. Thinking back to your analysis class and convergence proofs for sequences, you used this property whenever you chose $N = \max\{N_1, N_2, \dots\}$.

Ex
0. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all directed sets with their usual orders
1. Let X be a set and $\mathcal{F}$ be the collection of finite subsets of X. For $A, B \in \mathcal{F}$ write $A \leq B$ if $A \subset B$; this makes $\mathcal{F}$ into a directed set (Homework 4).
2. Let X be a topological space and fix $x \in X$. Let $\mathcal{N}(x)$ be the collection of neighborhoods of x. For $U, V \in \mathcal{N}(x)$ write $U \leq V$ iff $U \supset V$. (We say $\mathcal{N}(x)$ is ordered by reverse inclusion.) Then $\mathcal{N}(x)$ is a directed set:
   1. $U \supset U \Rightarrow U \leq U$ for all $U \in \mathcal{F}$
   2. $U \supset V$ and $V \supset W \Rightarrow U \supset W$ so that $U \leq W$.
   3. For $U, V \in \mathcal{F}$, $U \cap V$ is open and contains x, so $U, V \leq U \cap V$.
3. If I, J are directed sets, then $I \times J$ is a directed set under the relation $(i,j) \leq (i',j')$ iff $i \leq i'$ and $j \leq j'$. (Exercise check this) $\square$

- Note that the relation in the definition of directed set is close to but not quite the same as a partial order relation. What is missing?

Def Let X be a topological space. A <u>net</u> in X is a function $x: I \to X$ where I is a directed set. One usually writes $x_i := x(i)$ and denotes the net by $(x_i)_{i \in I}$.

- This notation should remind you of sequences, and indeed any sequence is a net:

Ex 1. Since $\mathbb{N}$ is a directed set, any sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ is a net in $\mathbb{R}$. More explicitly, $x: \mathbb{N} \to \mathbb{R}$ defined by $x(n) := x_n$ is a net.

2. Since $\mathbb{R}$ is a directed set, any function $f: \mathbb{R} \to \mathbb{R}$ is a net $(f(t))_{t \in \mathbb{R}}$ in $\mathbb{R}$.

3. Fix an interval $[a,b] \subset \mathbb{R}$. Let I be the collection of pairs (P,S) of finite sets of the form

$$P = \{x_0, x_1, x_2, \dots, x_n\}$$
$$S = \{t_1, t_2, \dots, t_n\}$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and $x_{j-1} \leq t_j \leq x_n$ for $j = 1, \dots, n$. In other words, P is a partition of $[a,b]$ and S consists of sample points. We can make I into a directed set by writing $(P,S) \leq (P',S')$ iff $P \subset P'$ and $S \subset S'$. (<u>Exercise</u> check this satisfies the definition of a directed set.) For $f: [a,b] \to \mathbb{R}$

$$\left(\sum_{j=1}^{n} f(t_j) \cdot (x_j - x_{j-1}) \right)_{(\{x_0, x_1, \dots, x_n\}, \{t_1, \dots, t_n\}) \in I}$$

is a net (of Riemann sums) in $\mathbb{R}$.

4. Let X be a topological space. Fix $x_0 \in X$ and let $\mathcal{N}(x_0)$ the collection of neighborhoods of x_0, which we've seen is a directed set when ordered by reverse inclusion. For each $U \in \mathcal{N}(x_0)$, pick an element $x_U \in U$. Then $(x_U)_{U \in \mathcal{N}(x_0)}$ is a net in X. $\square$

Def Let X be a topological space and let $x_0 \in X$ be a point. We say a net $(x_i)_{i \in I}$ in X <u>converges</u> to x_0 if for every neighborhood U of x_0 there exists $i_0 \in I$ so that $x_i \in U$ for all $i_0 \leq i$. In this case we say x is a <u>limit</u> of the net $(x_i)_{i \in I}$ and write $\lim_{i \to \infty} x_i = x_0$.

Ex Consider the nets in Examples 1 - 4 above. We discuss their convergence below

1. Let $\mathbb{R}$ have the standard topology. We claim a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ converges to $x_0 \in \mathbb{R}$ in the above sense iff for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$

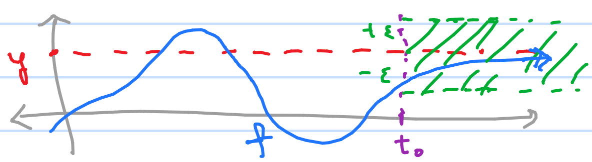
$$|x_n - x_0| < \varepsilon$$

(That is, this new notion of convergence agrees with the one you saw in analysis.) Indeed, if $(x_n)_{n\in\mathbb{N}}$ converges to $x_0 \in \mathbb{R}$, then $\forall \varepsilon > 0$ $(x_0 - \varepsilon, x_0 + \varepsilon)$ is a neighborhood of x_0. Thus there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, $x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)$. That is,

$$x_0 - \varepsilon < x_n < x_0 + \varepsilon \iff -\varepsilon < x_n - x_0 < \varepsilon \iff |x_n - x_0| < \varepsilon$$

Conversely, let U be a neighborhood of x_0. By Homework 3, $\exists \varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subset U$. Letting $n_0 \in \mathbb{N}$ correspond to this ε implies $|x_n - x_0| < \varepsilon$ for all $n \geq n_0$, which by the above is equivalent to $x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for all $n \geq n_0$. Thus $x_n \in U$ for all $n \geq n_0$ and therefore $(x_n)_{n\in\mathbb{N}}$ converges to x_0.

2. For $f: \mathbb{R} \to \mathbb{R}$, $(f(t))_{t\in\mathbb{R}}$ converges to some $y \in \mathbb{R}$ iff $\forall \varepsilon > 0$ there exists t_0 so that $t \geq t_0$ implies $|f(t) - y| < \varepsilon$. (This follows by the same reasoning as in the previous example). But this is just saying $f(t)$ has a horizontal asymptote at $+\infty$.



3. For $f: [a,b] \to \mathbb{R}$, the net

$$\left(\sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \right)_{(\{x_0, x_1, \dots, x_n\}, \{t_1, \dots, t_n\}) \in \mathcal{I}}$$

converges iff f is Riemann integrable, in which case the limit is

$$\int_a^b f(t)\,dt.$$

4. For $x_0 \in X$, the net $(x_U)_{U\in\mathcal{N}(x_0)}$ (where $x_U \in U$) converges to x_0. Indeed, let U be a neighborhood of x_0. Then for any $V \in \mathcal{N}(x_0)$ with $U \leq V$ we have

$$x_V \in V \subset U.$$

Thus $(x_U)_{U\in\mathcal{N}(x_0)}$ converges to x_0. $\square$

- Recall that our original motivation for considering nets was to give an alternate characterization of closed sets. We acheive this with the following theorem.

Thm Let X be a topological space. $A \subset X$ is closed if and only if for any convergent net $(x_i)_{i\in I} \subset A$ one has $\lim_{i\to\infty} x_i \in A$.

Proof ($\Longrightarrow$): Assume A is closed and suppose, towards a contradiction, that there exists a convergent net $(x_i)_{i\in I} \subset A$ with $x_0 := \lim_{i\to\infty} x_i \notin A$. So $x_0 \in X\setminus A$, which is a neighborhood of x since $X\setminus A$ is open. The definition of convergence implies $\exists\, i_0 \in I$ so that $x_i \in X\setminus A$ for all $i \geq i_0$. This contradicts $x_i \in A$ for all $i \in I$.

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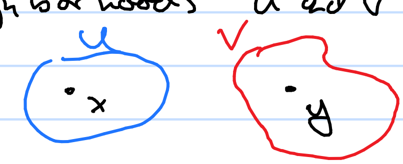
($\Longleftarrow$): We will show $X\setminus A$ is open. Note that if every $x \in X\setminus A$ has a neighborhood U_x satisfying $x \in U_x \subset X\setminus A$, then

$$X\setminus A = \bigcup_{x\in X\setminus A} U_x$$

is open. So suppose, towards a contradiction, that there exists $x \in X\setminus A$ such that every neighborhood U of x fails to be contained in $X\setminus A$. This means $U \cap A \neq \emptyset$ and so let $x_U \in U \cap A$. If $\mathcal{N}(x)$ denotes the directed set of neighborhoods of x (ordered by reverse inclusion), then we have seen the net $(x_U)_{U\in\mathcal{N}(x)}$ converges to x. But this net is contained in A and $x \notin A$, contradicting our hypothesis. $\square$

- The limit of a net $(x_i)_{i\in I} \subset X$ is not necessarily unique. This is because for $x_0 \in X$ there may be some $y_0 \in X$ so that $y_0 \in U$ for all neighborhoods U of x_0, and vice versa. For example consider a two element set $X = \{a, b\}$ equipped with the trivial topology $\mathcal{T} = \{\emptyset, X\}$. Then X is the only neighborhood of a or b, and so any net converging to a must also converge to b (and vice-versa). One way to avoid this situation is to only consider topological spaces of the following type.

Def A <u>Hausdorff space</u> is a topological space X where for any distinct $x, y \in X$ there exist disjoint neighborhoods U and V of x and y, respectively.



EX $\mathbb{R}$ with the standard topology is Hausdorff: if $x, y \in \mathbb{R}$ are distinct then for $\varepsilon = \frac{1}{2}|x-y|$ we have $U = (x-\varepsilon, x+\varepsilon)$ and $V = (y-\varepsilon, y+\varepsilon)$ are disjoint neighborhoods.
Recall that the lower-limit topology and K-topology are both finer than the standard topology. Thus $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are also Hausdorff. $\boxed{}$

Thm Let X be a Hausdorff space. Then any convergent net $(x_i)_{i\in I}\subset X$ has a unique limit.

Proof Let $(x_i)_{i\in I}$ be a convergent net with limit x_0. Let $y_0\in X$ be distinct from x_0. Since X is Hausdorff, there exist disjoint neighborhoods U and V of x_0 and y_0, respectively. Since $\lim_{i\to\infty} x_i = x_0$, there exists $i_0\in I$ so that $x_i\in U$ for all $i\geq i_0$. We claim that y_0 cannot also be a limit of $(x_i)_{i\in I}$. Indeed, if it was then there would exist $j_0\in I$ so that $x_i\in V$ for all $i\geq j_0$. Since I is a directed set, there exists $k\in I$ with $k\geq i_0$ and $k\geq j_0$. We would then have $x_k\in U$ and $x_k\in V$, contradicting $U\cap V=\phi$. Thus y_0 is not a limit of $(x_i)_{i\in I}$. Since $y_0\in X\setminus\{x_0\}$ was arbitrary, we see that x_0 is the unique limit. $\square$

- Another nice feature of Hausdorff spaces is that, like in $\mathbb{R}$, singleton sets are closed. Consequently finite sets are closed as finite unions of singleton sets.

Thm Finite subsets of Hausdorff spaces are closed.

Proof By the discussion preceding the theorem it suffices to show a singleton set is closed. Fix $x\in X$. For every $y\in X\setminus\{x\}$, there exists a neighborhood U_y of y that does not contain x. That is, $U_y\subset X\setminus\{x\}$. Thus

$$X\setminus\{x\} = \bigcup_{y\in X\setminus\{x\}} U_y$$

is open and therefore $\{x\}$ is closed. $\square$

- We can give an even faster proof of the above using nets: suppose $(x_i)_{i\in I}\subset\{x\}$ is a convergent net. Then $x_i = x$ for all $i\in I$ and so x is the unique limit of the net, and of course $x\in\{x\}$. Thus $\{x\}$ is closed.

Rem You might have noticed that we did not use the full definition of being Hausdorff in the above proof: we did not need that x had a neighborhood distinct from U_y. We say a space X is T_1 if for distinct $x,y\in X$ there exist neighborhoods $U\ni x$ and $V\ni y$ such that $y\notin U$ and $x\notin V$. Every Hausdorff space is T_1, but the converse is not true: define a topology on $\mathbb{R}$ by letting $U\subset\mathbb{R}$ be open iff $\mathbb{R}\setminus U$ is finite, then $\mathbb{R}$ is T_1 but not Hausdorff with this topology. The above theorem still holds for T_1 spaces and by the same proof, however it is not true that convergent nets have unique limits in T_1 spaces. $\square$

- Limit points also behave nicely in Hausdorff spaces. In fact, the following proof also works for T_1 spaces.

Thm Let X be a Hausdorff space. For $A \subset X$, $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points in A.

Proof ($\Longrightarrow$): Assume x is a limit point of A. Suppose, toward a contradiction, that there is a neighborhood U of x with $U \cap A$ finite. Then $U \cap (A \setminus \{x\})$ is also finite, say

$$U \cap (A \setminus \{x\}) = \{a_1, \dots, a_n\}.$$

By the previous theorem $\{a_1, \dots, a_n\}$ is closed and therefore

$$V := U \setminus \{a_1, \dots, a_n\} = U \cap (X \setminus \{a_1, \dots, a_n\})$$

is open. But then V is an open neighborhood of x (note $x \neq a_j$ for any $j = 1, \dots, n$ since $a_j \in A \setminus \{x\}$) with $V \cap (A \setminus \{x\}) = \emptyset$. This contradicts x being a limit point of A.

($\Longleftarrow$): Let U be a neighborhood of x. If $U \cap A$ is infinite, then so is $U \cap (A \setminus \{x\})$. In particular, this latter set is non-empty and thus x is a limit point of A. $\square$

- We conclude with a theorem summarizing how the Hausdorff condition interacts with the topics discussed in the previous three sections. We leave the proofs as exercises.

Thm
1. A set with an order relation and the order topology is Hausdorff.
2. A product of Hausdorff spaces is Hausdorff.
3. A subspace of a Hausdorff space is Hausdorff.

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§ 18 Continuous Functions

Def Let X and Y be topological spaces. We say a function $f: X \to Y$ is <u>continuous</u> if for each open set $V \subset Y$ the set $f^{-1}(V)$ is open in X.
For $x \in X$, we say f is <u>continuous at x</u> if for every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subset V$.

- Since the definition depends on the topologies on X and Y, we may sometimes need to say f is continuous <u>relative</u> to the topologies on X and Y.

Ex ① If X has the discrete topology, then all functions $f: X \to Y$ are continuous since $f^{-1}(V) \subset X$ is open for any subset $V \subset Y$ (not to mention the open ones).
If Y has the trivial topology, then all functions $f: X \to Y$ are continuous because the only open subsets of Y are ϕ and Y for which $f^{-1}(\phi) = \phi$ and $f^{-1}(Y) = X$ are open subsets of X.

② Let $f: \mathbb{R} \to \mathbb{R}$. Then f is continuous at $x \in \mathbb{R}$ in the above sense if and only if f satisfies the ε-δ definition of continuity at x from analysis:
($\Longrightarrow$): Let $\varepsilon > 0$. Note that $V := (f(x) - \varepsilon, f(x) + \varepsilon)$ is a neighborhood of $f(x)$. Consequently there exists a neighborhood U of x such that $f(U) \subset V$. By Exercise 1 on Homework 3, since U is open and $x \in U$ there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$. Consequently

$$f((x - \delta, x + \delta)) \subset f(U) \subset V = (f(x) - \varepsilon, f(x) + \varepsilon).$$

Finally, note that $x' \in (x - \delta, x + \delta)$ iff $|x - x'| < \delta$ and $f(x') \in (f(x) - \varepsilon, f(x) + \varepsilon)$ iff $|f(x) - f(x')| < \varepsilon$. Thus if $x' \in \mathbb{R}$ satisfies $|x - x'| < \delta$, then $|f(x) - f(x')| < \varepsilon$.
That is, f is ε-δ continuous at x.
($\Longleftarrow$): We leave this as an exercise.
It is also true that f is continuous iff f satisfies the ε-δ definition of continuity at every $x \in \mathbb{R}$. We also leave the proof of this as an exercise. □

Prop Let X and Y be topological spaces
① If $\mathcal{B}$ is a basis for the topology on Y, then $f: X \to Y$ is continuous if and only if $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$.
② If $\mathcal{S}$ is a subbasis for the topology on Y, then $f: X \to Y$ is continuous if and only if $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$.

Proof

1: If f is continuous, then $f^{-1}(B)$ is open for all $B \in \mathcal{B}$ since the basis sets are open in the topology on Y.

Conversely, suppose $f^{-1}(B)$ is open for all $B \in \mathcal{B}$. Let $U \subset Y$ be open. Recall that U can be written as a union of basis sets; namely

$$U = \bigcup_{\mathcal{B} \ni B \subset U} B.$$

Consequently,

$$f^{-1}(U) = \bigcup_{\mathcal{B} \ni B \subset U} f^{-1}(B)$$

Thus $f^{-1}(U)$ is open in X as a union of open sets. Since $U \subset Y$ was an arbitrary open set, f is continuous.

2: If f is continuous, then $f^{-1}(S)$ is open for all $S \in \mathcal{S}$ since the subbasis sets are open in the topology on Y.

Conversely, suppose $f^{-1}(S)$ is open for all $S \in \mathcal{S}$. Let $\mathcal{B}$ be the collection of all finite intersections of sets in $\mathcal{S}$. Thus for $B \in \mathcal{B}$ one has

$$B = S_1 \cap \cdots \cap S_n$$

for some $S_1, \ldots, S_n \in \mathcal{S}$. Consequently

$$f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$$

is open in X. Recall that $\mathcal{B}$ is a basis generating the same topology as $\mathcal{S}$ (namely the topology on Y). Thus by part 1, f is continuous. $\square$

EX: Let $\mathbb{R}$ and $\mathbb{R}_\ell$ denote the real numbers equipped with the standard and lower limit topologies, respectively. Consider

$$f : \mathbb{R}_\ell \to \mathbb{R} \qquad\qquad g : \mathbb{R} \to \mathbb{R}_\ell$$
$$x \mapsto x \qquad\qquad\qquad x \mapsto x$$

Then f is continuous while g is <u>not</u>. Indeed, $f^{-1}((a,b)) = (a,b)$ which is open in the lower limit topology. Since open intervals are a basis for the standard topology, the previous proposition implies f is continuous. On the other hand, $[0,1)$ is open in $\mathbb{R}_\ell$ but $g^{-1}([0,1)) = [0,1)$ is <u>not</u> open in $\mathbb{R}$. Hence g is <u>not</u> continuous. $\square$

- The above example really just comes down to the fact that the lower limit topology is finer than the standard topology.

- Using the symmetry between open and closed sets we can characterize continuity using closed sets:

Thm Let $f: X \to Y$ be a function between topological spaces. Then the following are equivalent:

1. f is continuous
2. For every subset $A \subset X$, one has $f(\bar{A}) \subset \overline{f(A)}$.
3. For every closed subset $B \subset Y$, the set $f^{-1}(B)$ is closed in X.
4. For each $x \in X$, f is continuous at x.

Proof We'll show 1 $\Rightarrow$ 2 $\Rightarrow$ 3 $\Rightarrow$ 1 and 1 $\Rightarrow$ 4 $\Rightarrow$ 1.

1 $\Rightarrow$ 2: Assume f is continuous and let $A \subset X$. Let $x \in \bar{A}$. In order to show $f(x) \in \overline{f(A)}$, we must show every neighborhood of $f(x)$ intersects $f(A)$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x. Since $x \in \bar{A}$, we know there exists $a \in A \cap f^{-1}(V)$. Hence $f(a) \in f(A) \cap V$, so V intersects $f(A)$.

2 $\Rightarrow$ 3: Let $B \subset Y$ be closed. Set $A := f^{-1}(B)$. To show A is closed we will show $A = \bar{A}$. Since $A \subset \bar{A}$ always, it suffices to show $\bar{A} \subset A$. Let $x \in \bar{A}$. Then since $f(A) \subset B$, we have by 2

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B.$$

Thus $f(x) \in B$ and so $x \in f^{-1}(B) = A$. Hence $\bar{A} \subset A$.

3 $\Rightarrow$ 1: Let $V \subset Y$ be open. We must show $f^{-1}(V)$ is open in X. Let $B := Y \setminus V$, which is a closed set. Consequently, the following set is closed in X:

$$f^{-1}(B) = f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V).$$

This set being closed in X implies $f^{-1}(V)$ is open in X, as desired.

1 $\Rightarrow$ 4: Fix $x \in X$ and let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open and contains x, hence is a neighborhood of x. Since $f(f^{-1}(V)) \subset V$, we see that f is continuous at x.

4 $\Rightarrow$ 1: Let $V \subset Y$ be open. For each $x \in f^{-1}(V)$, V is a neighborhood of $f(x)$ and so continuity at f implies there is a neighborhood U_x of x satisfying $f(U_x) \subset V$. Thus $U_x \subset f^{-1}(V)$. Consequently

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x,$$

so $f^{-1}(V)$ is open and f is continuous. $\square$

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- Recall from analysis that an equivalent condition for $f: \mathbb{R} \to \mathbb{R}$ to be continuous at $x \in \mathbb{R}$ was the following: whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence converging to x, the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$. This holds for general topological spaces too, so long as we replace "sequences" with "nets."

Thm Let $f: X \to Y$ be a function between topological spaces and let $x \in X$. Then f is continuous at x if and only if whenever $(x_i)_{i\in I} \subset X$ is a net converging to x, the net $(f(x_i))_{i\in I} \subset Y$ converges to $f(x)$.

Proof ($\Rightarrow$): Assume f is continuous at x and suppose $(x_i)_{i\in I} \subset X$ is a net converging to x. Let V be a neighborhood of $f(x)$. Then there exists a neighborhood U of x such that $f(U) \subset V$. By definition of convergence there exists $i_0 \in I$ so that $x_i \in U$ for all $i \geq i_0$. Consequently $f(x_i) \in f(U) \subset V$ for all $i \geq i_0$. Thus $(f(x_i))_{i\in I}$ converges to $f(x)$.

($\Leftarrow$): We will proceed by contrapositive. Assume f is not continuous at x. Then there exists a neighborhood V of $f(x)$ such that for all neighborhoods U of x one has $f(U) \not\subset V$. Let $\mathcal{N}$ be the collection of neighborhoods of x (which we make into a directed set via reverse inclusion). For each $U \in \mathcal{N}$, let $x_U \in U$ be such that $f(x_U) \notin V$. Then — as we have seen before — the net $(x_U)_{U\in\mathcal{N}}$ converges to x. However, $(f(x_U))_{U\in\mathcal{N}}$ does <u>not</u> converge to $f(x)$ since it lies entirely outside of V. $\square$

Def Let X and Y be topological spaces. A <u>homeomorphism</u> is a bijection $f: X \to Y$ such that f and f^{-1} are continuous. In this case we say X and Y are <u>homeomorphic</u>.

- Observe that if $f: X \to Y$ is a homeomorphism, then $U \subset X$ is open iff $f(U) \subset Y$ is open. Indeed, since f is a bijection we have

$$f^{-1}(f(U)) = U \qquad \text{and} \qquad (f^{-1})^{-1}(U) = f(U)$$

The former shows U is open when $f(U)$ is open (by continuity of f), and the latter shows $f(U)$ is open when U is open (by continuity of f^{-1}). Similarly, $V \subset Y$ is open iff $f^{-1}(V) \subset X$ is open. Thus a homeomorphism maps the topology $\mathcal{T}$ on X to the topology $\mathcal{T}'$ on Y in a bijective way:

$$\begin{aligned} \mathcal{T} &\longleftrightarrow \mathcal{T}' \\ U &\longmapsto f(U) \\ f^{-1}(V) &\longmapsfrom V \end{aligned}$$

One consequence of this is that if X has a property that can be expressed using only the topology on X, then any Y homeomorphic to X will have the same property. For example, if X is Hausdorff then so is Y.

Def Let X and Y be topological spaces. A <u>topological embedding</u> of X into Y is a injective map $f: X \to Y$ such that the restriction of f to its range $f': X \to f(X)$

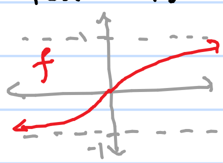
is a homeomorphism.

$\boxed{\text{Ex}}$ 0. For any topological space X, the identity map $X \ni x \mapsto x$ is a homeomorphism. For any subspace $Y \subset X$, the inclusion map $Y \ni y \mapsto y \in X$ is a topological embedding.

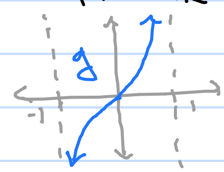
1. If X and Y both have the discrete topology, then they are homeomorphic iff they have the same cardinality. Likewise if they have the trivial topology. (Exercise check these)
What if X has the discrete topology and Y has the trivial topology?

2. Consider the functions $f : \mathbb{R} \to (-1,1)$ and $g : (-1,1) \to \mathbb{R}$

$$f(x) = \frac{x}{1+|x|}$$



$$g(x) = \frac{x}{1-|x|}$$



Then $g = f^{-1}$ (Exercise check this) so that f is a bijection. We further claim f is a homeomorphism when $\mathbb{R}$ has the standard topology and $(-1,1) \subset \mathbb{R}$ has the subspace topology.
To see this recall that open intervals (a,b) form a basis for the standard topology and consequently $(a,b) \cap (-1,1)$ form a basis for the subspace topology. But such intersections are either empty or of the form (c,d) for $-1 \le c < d \le 1$. Thus continuity of f and f^{-1} will follow, by the first proposition in this section, if we can show they map open intervals to open intervals.
Observe for $x < y$ we have

$$\frac{x}{1+|x|} < \frac{y}{1+|y|} \iff x(1+|y|) < y(1+|x|)$$

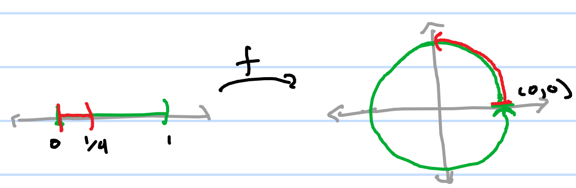
$$\iff 0 < y - x + y|x| - x|y|$$

$$\iff \begin{cases} 0 < y - x + |x||y|\left(\frac{y}{|y|} - \frac{x}{|x|}\right) & \text{if } x \neq 0 \neq y \\ 0 < y - x & \text{otherwise} \end{cases}$$

✓

Thus $x < y$ implies $f(x) < f(y)$ and consequently $f((a,b)) = (f(a), f(b))$.
Similarly for f^{-1} and so f is a homeomorphism.

3. Let $S^1 \subset \mathbb{R}^2$ denote the unit circle. Then $f : [0,1) \to S^1$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is a bijection, but it is not a homeomorphism.
Indeed, the set $[0, \frac{1}{4})$ is open in $[0,1)$, but $f([0,\frac{1}{4}))$ is not open in S^1 because $f(0) = (0,0)$ has no neighborhood contained in $f([0,\frac{1}{4}))$.



$\square$

Constructing Continuous Functions

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- Since the definition of continuity we are using is the same as the ε-δ definition for functions $f: \mathbb{R} \to \mathbb{R}$, your analysis class is a rich source of examples of continuous functions. Another source comes from applying the following constructions to existing examples on more general topological spaces.

Thm (Rules for constructing continuous functions)
Let X, Y, and Z be topological spaces.

1. Constant Function: If $f: X \to Y$ sends all $x \in X$ to a single $y_0 \in Y$, then f is continuous.
2. Inclusion: If $A \subset X$ is a subspace, then the inclusion function $i: A \to X$ defined by $i(x) = x$ is continuous.
3. Composites: If $f: X \to Y$ and $g: Y \to Z$ are continuous, then $g \circ f: X \to Z$ is continuous.
4. Restricting the Domain: If $f: X \to Y$ is continuous and if $A \subset X$ is a subspace, then the restriction $f|_A: A \to Y$ is continuous.
5. Restricting or Expanding the Range: Let $f: X \to Y$ be continuous. If $f(x) \subset B \subset Y \subset Z$, then the functions
$$g: X \to B \qquad h: X \to Z$$
obtained from f by restricting and expanding the range, respectively, are continuous.
6. Local Continuity: The map $f: X \to Y$ is continuous if X can be written as the union of a collection $\mathcal{S}$ of open sets such that $f|_U$ is continuous for every $U \in \mathcal{S}$.

Proof

1: For $V \subset Y$ open, we have $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \emptyset$ otherwise. In either case $f^{-1}(V)$ is open and so f is continuous.

2: For $U \subset X$ open, we have $i^{-1}(U) = A \cap U$, which is open in A.

3: For $W \subset Z$ open, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since g is continuous, $g^{-1}(W)$ is open in Y. Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X.

4: $f|_A = f \circ i$ where $i: A \to X$ is the inclusion map. Thus $f|_A$ is continuous by 2 and 3.

5: If $V \subset B$ is open in B, then there exists $W \subset Y$ such that $V = B \cap W$. Since $f(x) \subset B$ we have $f^{-1}(B) = X$ and so
$$g^{-1}(V) = f^{-1}(V) = f^{-1}(B \cap W) = f^{-1}(B) \cap f^{-1}(W) = X \cap f^{-1}(W) = f^{-1}(W).$$
which is open since f is continuous. Thus g is continuous.
Observe that $h = i \circ f$ where $i: Y \to Z$ is the inclusion map. So h is continuous by 2 and 3.

(6) : Let $\mathcal{S}$ be a collection of open sets satisfying $\bigcup_{U \in \mathcal{S}} U = X$ and for each $U \in \mathcal{S}$, $f|_U$ is continuous. Let $V \subset Y$ be open. Then

$$f^{-1}(V) = \bigcup_{U \in \mathcal{S}} U \cap f^{-1}(V) = \bigcup_{U \in \mathcal{S}} (f|_U)^{-1}(V)$$

Each $(f|_U)^{-1}(V)$ is open by the continuity of $f|_U$. Thus $f^{-1}(V)$ is open and so f is continuous. $\square$

- There is a version of 6 in the previous theorem where X is instead decomposed into a finite union of closed sets.

Thm (Pasting Lemma)

Let $X = A \cup B$ for closed subsets $A, B \subset X$. If $f: A \to Y$ and $g: B \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then $h: X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof Note that h is well-defined by $f|_{A\cap B} = g|_{A \cap B}$. Let $C \subset Y$ be a closed set. Then

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous, $f^{-1}(C)$ is closed in A, but A is closed in X and so $f^{-1}(C)$ is closed in X. Similarly, $g^{-1}(C)$ is closed in X. Hence $h^{-1}(C)$ is a finite union of closed sets and consequently is closed. By the first theorem in this section we see that h is continuous. $\square$

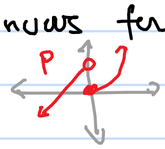
EX Since $\mathbb{R} = (-\infty, 0] \cup [0, +\infty)$ and these rays are closed, if $f: (-\infty, 0] \to \mathbb{R}$ and $g: [0, +\infty) \to \mathbb{R}$ are continuous then we can use the previous theorem to build a continuous function $h: \mathbb{R} \to \mathbb{R}$, so long as $f(0) = g(0)$.

So $f(x) = x$ and $g(x) = x^2$ paste together to produce the continuous function

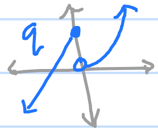
$$h(x) = \begin{cases} x & \text{if } x \leq 0 \\ x^2 & \text{if } x \geq 0 \end{cases} = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

However, $f(x) = x+1$ and $g(x) = x^2$ cannot be pasted together into a well-defined function. Moreover, using $f|_{(-\infty,0)}$ or $g|_{(0,+\infty)}$ instead fails to produce a continuous function:

$$p(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$



$$q(x) \begin{cases} x+1 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$



are both not continuous. $\square$

- **Exercise** Use induction to show if $X = A_1 \cup \cdots \cup A_n$ for closed sets A_j and if $f_j : A_j \to Y$ are continuous with $f_i(x) = f_j(x)$ for all $x \in A_i \cap A_j$, then there exists a continuous $h : X \to Y$ with $h|_{A_j} = f_j$ for each $j = 1, \ldots, n$. Can this be done for an infinite collection of closed sets?

- We conclude this section with a characterization of continuity for functions whose range is a product of topological spaces. Unfortunately, there is no similar result when the domain is a product.

Thm Let $f : X \to Y \times Z$ be a function defined by

$$f(x) = (f_1(x), f_2(x))$$

for $f_1 : X \to Y$ and $f_2 : X \to Z$. Then f is continuous iff f_1 and f_2 are continuous.

Proof Let $\pi_1 : Y \times Z \to Y$ and $\pi_2 : Y \times Z \to Z$ be the coordinate projections. These maps are continuous: if $V \subset Y$ is open then

$$\pi_1^{-1}(V) = V \times Z$$

is open in $Y \times Z$. Similarly for π_2.

Now suppose f is continuous. Then $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ are continuous. Conversely, suppose f_1 and f_2 are continuous. To show f is continuous, recall that it suffices to show $f^{-1}(B)$ is open for $B \in \mathcal{B}$ where $\mathcal{B}$ is some basis for the topology on $Y \times Z$. Recall that

$$\mathcal{B} = \{ V \times W \mid V \subset Y, W \subset Z \text{ open} \}$$

is a basis for the product topology on $Y \times Z$. Fix $V \times W \in \mathcal{B}$ and observe that $x \in f^{-1}(V \times W)$ iff

$$(f_1(x), f_2(x)) = f(x) \in V \times W$$

iff $f_1(x) \in V$ and $f_2(x) \in W$ iff $x \in f_1^{-1}(V) \cap f_2^{-1}(W)$. That is

$$f^{-1}(V \times W) = f_1^{-1}(V) \cap f_2^{-1}(W).$$

This is open since $f_1^{-1}(V)$ and $f_2^{-1}(W)$ are both open. Hence f is continuous. $\square$

- Note that any $f : X \to Y \times Z$ has the form $f(x) = (f_1(x), f_2(x))$ by defining $f_1 := \pi_1 \circ f$ and $f_2 := \pi_2 \circ f$. These are called the coordinate functions of f.

Ex A continuous curve in $\mathbb{R}^2$ is the image of a continuous function $f : [0,1] \to \mathbb{R}^2$. The theorem implies there are continuous coordinate functions $f_1, f_2 : [0,1] \to \mathbb{R}$ such that $f(t) = (f_1(t), f_2(t))$ for all $t \in [0,1]$. $\square$

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§19 The Product Topology (part 2)

- Back in §15 we consider only a product of two topological spaces $X \times Y$. In this section we will consider a product of arbitrarily many topological spaces (including an uncountable infinity of them). There are two somewhat natural topologies one can define on such products (though only one will be dubbed the product topology), and to better illustrate the difference we first consider the case of countable products:

Ex For each $n \in \mathbb{N}$, let A_n be a topological space and consider
$$X := A_1 \times A_2 \times \cdots \times A_N = \{ (x_1, x_2, \dots, x_N) \mid x_j \in A_j \ \ j = 1, \dots, N\}$$
for some $N \in \mathbb{N}$, and
$$Y := A_1 \times A_2 \times \cdots = \{ (x_j)_{j \in \mathbb{N}} \mid x_j \in A_j \ \ j \in \mathbb{N}\}$$
We can generate topologies on X and Y using either a basis or a subbasis.
The collection of subsets $U_1 \times \cdots \times U_N$ for $U_j \subset A_j$ open forms a basis for a topology on X, while the collection of $U_1 \times U_2 \times \cdots$ for $U_j \subset A_j$ open forms a a basis for a topology on Y. We call the generated topologies box topologies (recall $(a,b) \times (c,d) \subset \mathbb{R}^2$ was a box).
To form a subbasis, let π_j denote the coordinate projections for X and Y. Note that for $U \subset A_j$ we have in X
$$\pi_j^{-1}(U) = A_1 \times \cdots \times A_{j-1} \times U \times A_{j+1} \times \cdots \times A_N$$
while in Y
$$\pi_j^{-1}(U) = A_1 \times \cdots \times A_{j-1} \times U \times A_{j+1} \times \cdots$$
The collections
$$\bigcup_{j=1}^{N} \{\pi_j^{-1}(U) \mid U \subset A_j \text{ open}\} \quad \text{and} \quad \bigcup_{j \in \mathbb{N}} \{\pi_j^{-1}(U) \mid U \subset A_j\}$$
form subbases for topologies on X and Y, respectively, which we call product topologies.
Note that any finite intersection of the subbasis for X is of the form
$$\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \cdots \cap \pi_N^{-1}(U_N) = U_1 \times U_2 \times \cdots \times U_N.$$
Thus the subbasis generates the basis for the box topology on X. Hence the box and product topologies for finite products are the same.
This is not true for Y. Indeed, any finite intersection of elements in the subbasis is of the form
$$\pi_{j_1}^{-1}(U_1) \cap \cdots \cap \pi_{j_d}^{-1}(U_d) = A_1 \times \cdots A_{j_1 - 1} \times U_1 \times A_{j_1+1} \cdots \times A_{j_d - 1} \times U_d \times A_{j_d+1} \times \cdots$$
So all but finitely many of the factors in the above product are the full set A_j.
This shows the box topology can be strictly finer than the product topology on infinite products. $\square$

- To accommodate arbitrary products, we need some new notation and terminology

Def Let X be a set. For an index set J, a J-tuple in X is a function $x: J \to X$. For $j \in J$ we denote $x_j := x(j)$ and write $(x_j)_{j \in J}$ in place of x. We let X^J denote the set of all J-tuples in X.

- If $J = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, then a J-tuple is simply an n-tuple: $(x_j)_{j \in J} = (x_1, x_2, \ldots, x_n)$. If $J = \mathbb{N}$, then a J-tuple is a sequence.

Def Let $\{A_j \mid j \in J\}$ be a family of sets indexed by a set J. Set $X = \bigcup_{j \in J} A_j$. The cartesian product of this family is

$$\prod_{j \in J} A_j := \{(x_j)_{j \in J} \in X^J \mid x_j \in A_j \text{ for all } j \in J\}$$

- If $A_j = X$ for all $j \in J$, then $\prod_{j \in J} A_j = X^J$.

- Depending on the situation, we may write elements of $\prod_{j \in J} A_j$ as J-tuples or as functions $J \to X$.

- Let $\{X_j \mid j \in J\}$ be a family of topological spaces indexed by some set J. Consider the following collection of subsets of $\prod_{j \in J} X_j$:

$$\mathcal{B} = \{\prod_{j \in J} U_j \mid U_j \subset X_j \text{ open for all } j \in J\}$$

  Then $\mathcal{B}$ is a basis for a topology on $\prod_{j \in J} X_j$. Indeed X_j is open in X_j for all $j \in J$, hence $\prod_{j \in J} X_j \in \mathcal{B}$. This shows $\mathcal{B}$ satisfies 1 in the definition of a basis. For 2 observe

$$\left(\prod_{j \in J} U_j\right) \cap \left(\prod_{j \in J} V_j\right) = \prod_{j \in J} (U_j \cap V_j)$$

  Thus $\mathcal{B}$ is a basis.

Def Let $\{X_j \mid j \in J\}$ be a family of topological spaces indexed by some set J. The box topology on the cartesian product $\prod_{j \in J} X_j$ is the topology generated by the basis

$$\mathcal{B} := \{\prod_{j \in J} U_j \mid U_j \subset X_j \text{ open}\}$$

- The fact that we do not call the above topology "the product topology" (we reserve that name for a topology defined below) indicates that we have a certain amount of bias against it. One reason for this bias is the following example. We will see additional reasons later in the section (and the course).

$\boxed{\text{EX}}$ Consider $J = \mathbb{N}$ and $X_n = \mathbb{R}$ for all $n \in \mathbb{N}$. Then $\prod_{n \in \mathbb{N}} X_n = \mathbb{R}^{\mathbb{N}}$ sequences of real numbers. Equip $\mathbb{R}^{\mathbb{N}}$ with the box topology, where each $X_n = \mathbb{R}$ has the standard topology. Define a function $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ by

$$f(t) = (t, t, t, \dots)$$

We claim this function is <u>not</u> continuous. Indeed, for $U_n \subset \mathbb{R}$ open, $n \in \mathbb{N}$, observe that

$$f^{-1}\left(\prod_{n \in \mathbb{N}} U_n\right) = \left\{ t \in \mathbb{R} \mid (t, t, t, \dots) \in \prod_{n \in \mathbb{N}} U_n \right\} = \bigcap_{n \in \mathbb{N}} U_n .$$

However, letting $U_n = (-\frac{1}{n}, \frac{1}{n})$ for $n \in \mathbb{N}$ we have

$$\bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} (-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\},$$

which is not open in $\mathbb{R}$. Hence f is <u>not</u> continuous. $\square$

- The trouble we ran into in the above example is that an <u>infinite</u> intersection of open sets need not be open. However, if all but finitely many of the $U_n = \mathbb{R}$, then the infinite intersection reduces to a finite intersection of open sets, which is still open. This motivates our definition of the product topology below.

$\boxed{\text{Def}}$ Let $\{X_j \mid j \in J\}$ be a family of topological spaces indexed by some set J. For each $j_0 \in J$, define

$$\pi_{j_0} : \prod_{j \in J} X_j \to X_{j_0}$$

by $\pi_{j_0}((x_j)_{j \in J}) = x_{j_0}$. We call this map the <u>j_0 coordinate projection.</u>

- Observe that for $U \subset X_{j_0}$,

$$\pi_{j_0}^{-1}(U) = \left\{ (x_j)_{j \in J} \in \prod_{j \in J} X_j \mid x_{j_0} \in U \right\}$$

In particular, if $J = \mathbb{N}$ then

$$\pi_n^{-1}(U) = X_1 \times \cdots \times X_{n-1} \times U \times X_{n+1} \times \cdots$$

So all but finitely many factors are the full set X_j.

- For each $j \in J$, define $S_j := \{ \pi_j^{-1}(U) \mid U \subset X_j \text{ open} \}$ and let $S := \bigcup_{j \in J} S_j$.
Since X_j is open in X_j

$$\prod_{k \in J} X_k = \pi_j^{-1}(X_j) \in S_j$$

for all $j \in J$. Consequently $\mathcal{S}$ is a subbasis for a topology on $\prod_{j\in J} X_j$. (In fact, each $\mathcal{S}_j$ is a subbasis but we won't consider the topology it generates right now.)

Def Let $\{X_j \mid j \in J\}$ be a family of topological spaces indexed by some set J. The product topology on the Cartesian product $\prod_{j\in J} X_j$ is the topology generated by the subbasis $\mathcal{S} = \bigcup_{j\in J} \mathcal{S}_j$ where

$$\mathcal{S}_j = \{ \pi_j^{-1}(U) \mid U \subset X_j \text{ open} \}.$$

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- Let $\mathcal{B}'$ be the collection of all finite intersections of sets in $\mathcal{S}$ (i.e. the basis that generates the product topology). Since $\pi_j^{-1}(U) \cap \pi_j^{-1}(V) = \pi_j^{-1}(U \cap V)$, $\mathcal{B}'$ consists of sets of the form

$$\pi_{j_1}^{-1}(U_1) \cap \cdots \cap \pi_{j_d}^{-1}(U_d)$$

for $d \in \mathbb{N}$, distinct $j_1, \ldots, j_d \in J$ and $U_{j_k} \subset X_{j_k}$ open. This intersection equals the cartesian product $\prod_{j\in J} U_j$ where $U_j = X_j$ for all but finitely many j (namely $j \in \{j_1, \ldots, j_d\}$). Consequently, if J is finite then the box and product topologies are the same.

Ex Equip $\mathbb{R}^{\mathbb{N}}$ with the product topology, where each $\mathbb{R}$ has the standard topology. Then $f: \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$f(t) = (t, t, t, \ldots)$$

is continuous. Indeed, it suffices to show $f^{-1}(B)$ is open in $\mathbb{R}$ for B in the basis $\mathcal{B}'$ described above. That is,

$$B = \pi_{n_1}^{-1}(U_1) \cap \cdots \cap \pi_{n_d}^{-1}(U_d)$$

for $d \in \mathbb{N}$, distinct $n_1, \ldots, n_d \in \mathbb{N}$, and $U_{n_k} \subset \mathbb{R}$ open. In this case,

$$f^{-1}(B) = U_1 \cap \cdots \cap U_d$$

is open as a finite intersection of open sets. Hence f is continuous. $\square$

Thm Let $\{X_j \mid j \in J\}$ be a family of topological spaces indexed by some set J. Then the box topology on $\prod_{j\in J} X_j$ is finer than the product topology. When J is finite, the two topologies are equal.

Proof The basis $\mathcal{B}$ for the box topology consists of cartesian products $\prod_{j\in J} U_j$ where $U_j \subset X_j$ is open for all $j \in J$. The product topology has a basis $\mathcal{B}'$ consisting of finite intersections of sets in the subbasis $\mathcal{S}$, and so $\mathcal{B}'$ consists of cartesian products $\prod_{j\in J} U_j$ where $U_j \subset X_j$ is open for all $j \in J$, but $U_j = X_j$ for all but finitely many $j \in J$. Thus $\mathcal{B}' \subset \mathcal{B}$, which implies the box topology is finer than the product topology. If J is finite, then $\mathcal{B}' = \mathcal{B}$ and so the topologies agree. $\square$

- Let's do a quick sanity check for the above theorem. We saw that

$$f: \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$$
$$t \mapsto (t, t, t, \dots)$$

is continuous for the product topology but not the box topology. Since continuity requires $f^{-1}(U)$ being open for all $U \subset \mathbb{R}^{\mathbb{N}}$ open, having more open sets (i.e. having a finer topology) in the range should make it harder to be continuous. Thus the above example is in line with the previous theorem.

Rem From now on, all Cartesian products $\prod_{j \in J} X_j$ will be given the product topology, unless we specifically state otherwise.

Thm Let $\{X_j \mid j \in J\}$ be a family of topological spaces. If $\mathcal{B}_j$ is a basis for the topology on X_j, then

$$\mathcal{B} = \{ \prod_{j \in J} B_j \mid B_j \in \mathcal{B}_j \text{ for each } j \in J\}$$

is a basis for the box topology on $\prod_{j \in J} X_j$ while

$$\mathcal{B}' = \{ \prod_{j \in J} B_j \mid B_j \in \mathcal{B}_j \cup \{X_j\} \text{ for each } j \in J \text{ and } B_j = X_j \text{ for all but finitely many } j \in J\}$$

is a basis for the product topology.

- We leave the proof of the above as an exercise.

Ex A basis for the box topology on $\mathbb{R}^{\mathbb{N}}$ is the collection of products of open intervals

$$(a_1, b_1) \times (a_2, b_2) \times \cdots$$

while sets of the form

$$\mathbb{R} \times \cdots \times \mathbb{R} \times (a_{n_1}, b_{n_1}) \times \mathbb{R} \times \cdots \times \mathbb{R} \times (a_{n_k}, b_{n_k}) \times \mathbb{R} \times \cdots$$

form a basis for the product topology on $\mathbb{R}^{\mathbb{N}}$. $\square$

Thm Let $\{X_j \mid j \in J\}$ be a family of topological spaces and let $A_j \subset X_j$ be a subset for each $j \in J$. Giving each $A_j \subset X_j$ the subspace topology and $\prod_{j \in J} A_j$ the box topology is the same as giving $\prod_{j \in J} X_j$ the box topology and $\prod_{j \in J} A_j$ the subspace topology. The same statement holds after replacing each instance of "box" with "product."

Proof We prove only the first statement as the second follows by adding "for at most finitely many $j \in J$" in the appropriate places. Observe that the collection of sets of the form

$$\prod_{j \in J} (A_j \cap U_j) = \left(\prod_{j \in J} A_j\right) \cap \left(\prod_{j \in J} U_j\right)$$

where $U_j \subset X_j$ is open for each $j \in J$, is a common basis for the two topologies. $\square$

Thm Let X_j be a Hausdorff space for each $j \in J$. Then $\prod_{j \in J} X_j$ is a Hausdorff space under both the box and product topologies.

Proof Let $(x_j), (y_j) \in \prod_{j \in J} X_j$ be distinct points. We must find disjoint neighborhoods for these points. Since the box topology is finer than the product topology, it suffices to find neighborhoods that are open in the product topology. Since $(x_j) \neq (y_j)$, $\exists\, j_0 \in J$ such that $x_{j_0}, y_{j_0} \in X_{j_0}$ are distinct. Since X_{j_0} is Hausdorff, there exists disjoint neighborhoods U and V of x_{j_0} and y_{j_0}, respectively. But then $\pi_{j_0}^{-1}(U)$ and $\pi_{j_0}^{-1}(V)$ are neighborhoods of (x_j) and (y_j), respectively, and

$$\pi_{j_0}^{-1}(U) \cap \pi_{j_0}^{-1}(V) = \pi_{j_0}^{-1}(U \cap V) = \pi_{j_0}^{-1}(\emptyset) = \emptyset.$$

Thus the neighborhoods are disjoint and so $\prod_{j \in J} X_j$ is Hausdorff. $\square$

Thm Let $\{X_j \mid j \in J\}$ be an indexed family of topological spaces, and for each $j \in J$ let $A_j \subset X_j$ be a subset. Then in both the box and product topologies on $\prod_{j \in J} X_j$ one has

$$\overline{\prod_{j \in J} A_j} = \prod_{j \in J} \overline{A_j}.$$

Proof We will prove the equality by proving the inclusions "$\subset$" and "$\supset$". We begin with the latter: let $(x_j) \in \prod_{j \in J} \overline{A_j}$. Let $U = \prod_{j \in J} U_j$ be a basis set for either the box or product topology (i.e. $U_j \subset X_j$ is open and if we use the product topology then $U_j = X_j$ for all but finitely many $j \in J$) that contains (x_j). Then for each $j \in J$, U_j is a neighborhood of $x_j \in \overline{A_j}$. Consequently $\exists\, y_j \in U_j \cap A_j$. It follows that

$$(y_j)_{j \in J} \in U \cap \prod_{j \in J} A_j.$$

So U intersects $\prod_{j \in J} A_j$. Since U was an arbitrary basis element containing (x_j), we have $(x_j) \in \overline{\prod A_j}$ by the second theorem in §17. Hence $\prod \overline{A_j} \subset \overline{\prod A_j}$.

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Conversely, let $(x_j) \in \overline{\prod A_j}$. We will show $x_j \in \overline{A_j}$ for all $j \in J$. Fix $j \in J$ and let $V \subset X_j$ be a neighborhood of x_j. Then $\pi_j^{-1}(V)$ is a neighborhood of (x_j) (in both topologies). Thus there exists $(y_j) \in \pi_j^{-1}(V) \cap \prod A_j$. Then

$$y_j = \pi_j((y_j)) \in V \cap A_j$$

So V intersects A_j, and the second theorem in §17 implies $x_j \in \overline{A_j}$. Hence $(x_j) \in \prod \overline{A_j}$ which implies the remaining inclusion and finishes the proof. $\square$

- While the previous three theorems applied to both the box and product topologies, the next theorem only applies to the product topology. In fact, we have already seen a counterexample for the box topology...

Thm Let X be a topological space and let $\{Y_j \mid j\in J\}$ be an indexed family of topological spaces. Give $\prod_{j\in J} Y_j$ the product topology. Let $f: X\to \prod_{j\in J} Y_j$ have the formula

$$f(x) = (f_j(x))_{j\in J}$$

where $f_j: X\to Y_j$ for each $j\in J$. Then f is continuous if and only if f_j is continuous for all $j\in J$.

Proof We first claim each coordinate projection $\pi_{j_0}: \prod_{j\in J} Y_j \to Y_{j_0}$ is continuous. This is immediate since for $V\subset Y_{j_0}$ open, $\pi_{j_0}^{-1}(V)$ is open since it is part of the subbasis defining the product topology.

Now, suppose f is continuous. Then for $j\in J$, $f_j = \pi_j\circ f$ is continuous as the composition of continuous functions. Conversely, assume f_j is continuous for each $j\in J$. Let $U = \prod U_j$ be a basis set for the product topology: $U_j\subset X_j$ is open for all $j\in J$ and $U_j = X_j$ for all but finitely many $j\in J$, say $\{j_1,\dots,j_d\}$. Then

$$\begin{aligned} f^{-1}(U) &= \{x\in X \mid f(x)\in \textstyle\prod U_j\} \\ &= \{x\in X \mid f_j(x)\in U_j \text{ for all } j\in J\} \\ &= \bigcap_{j\in J} f_j^{-1}(U_j). = f_{j_1}^{-1}(U_{j_1})\cap\cdots\cap f_{j_d}^{-1}(U_{j_d}) \end{aligned}$$

which is open. Hence f is continuous. $\square$

- We call the f_j in the previous theorem the coordinate functions of f. Note that any function whose range is a Cartesian product has coordinate functions by composing with the coordinate projections.

- For $f: \mathbb{R}\to\mathbb{R}^{\mathbb{N}}$ defined by $f(t) = (t,t,\dots)$, the coordinate functions are $f_n(t) = t$ for all $n\in\mathbb{N}$, which are continuous. We saw f was not continuous when $\mathbb{R}^{\mathbb{N}}$ is given the box topology. Thus the above theorem does not hold for the box topology.

§20 The Metric Topology (part 1)

- In this section we will add another method for defining a topology to our repertoire (joining order, product, and subspace topologies). This new method will come from a notion of distance on abstract spaces called a "metric". Recall that previously we would think of a neighborhood of some $x \in X$ as the points "close" to x. With metrics we will be able to make this notion precise.

Def A metric on a set X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying

1. $d(x,y) \geq 0$ for all $x,y \in X$, and equality holds if and only if $x=y$.
2. $d(x,y) = d(y,x)$ for all $x,y \in X$.
3. For all $x,y,z \in X$

$$d(x,z) \leq d(x,y) + d(y,z) \qquad \text{(Triangle Inequality)}$$

In this case, $d(x,y)$ is called the distance between x and y.

Ex 1. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x,y) := |x-y|$. Then d is a metric:

1. $d(x,y) = |x-y| \geq 0$ and $|x-y| = 0$ iff $x=y$.
2. For $x,y \in \mathbb{R}$

$$d(x,y) = |x-y| = |-(x-y)| = |-x+y| = |y-x| = d(y,x)$$

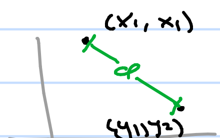
3. For $x,y,z \in \mathbb{R}$

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z)$$

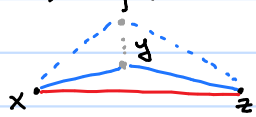
d is called the standard metric on $\mathbb{R}$.

2. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by the distance formula:

$$d((x_1,x_2),(y_1,y_2)) := \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$



Then d is a metric: 1 and 2 follow quickly from the definition while 3 is a fact from geometry — no side of a triangle is longer than the sum of the other two sides:



Similarly the distance formula for $\mathbb{R}^n$ defines a metric:

$$d((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)) = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2}$$

3. Let X be any set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{otherwise} \end{cases}$$

Then d is a metric: ① and ② are immediate from the definition of d. For ③, let $x, y, z \in X$. If $x = z$, then

$$d(x,z) = 0 \leq d(x,y) + d(y,z)$$

and we've done. Otherwise, $x \neq z$ and so we must have either $x \neq y$ or $y \neq z$. Consequently

$$d(x,z) = 1 \leq d(x,y) + d(y,z)$$

d is called the discrete metric on X. $\square$

Def Let X be a set with metric d. For $x \in X$ and $\varepsilon > 0$, the ε-ball centered at x is the set

$$B_d(x, \varepsilon) := \{ y \in X \mid d(x,y) < \varepsilon \}$$

we may just write $B(x,\varepsilon)$ if the metric d is clear from context.

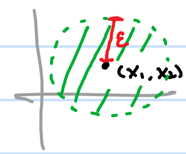
Ex We consider the metrics d in the previous examples.

① For $x \in \mathbb{R}$ and $\varepsilon > 0$

$$B_d(x,\varepsilon) = \{ y \in \mathbb{R} \mid |x-y| < \varepsilon \} = (x-\varepsilon, x+\varepsilon)$$

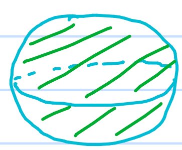
② For $(x_1, x_2) \in \mathbb{R}^2$ and $\varepsilon > 0$

$$B_d((x_1,x_2), \varepsilon) = \{ (y_1,y_2) \in \mathbb{R}^2 \mid \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} < \varepsilon \}$$



For $x \in \mathbb{R}^3$

$$B_d(x,\varepsilon) =$$



③ For X with discrete metric, if $x \in X$ and $\varepsilon > 0$

$$B_d(x,\varepsilon) = \begin{cases} x & \text{if } \varepsilon \leq 1 \\ X & \text{if } \varepsilon > 1. \end{cases}$$

$\square$

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- We will define a topology on a set equipped with a metric by using the ε-balls as a basis. We first require a lemma.

Lemma Let X be a set with metric d. For $x \in X$ and $\varepsilon > 0$, if $y \in B_d(x,\varepsilon)$ then there exists $\delta > 0$ so that $B_d(y,\delta) \subset B_d(x,\varepsilon)$.

Proof Since $y \in B_d(x,\varepsilon)$, we know $d(x,y) < \varepsilon$. Thus

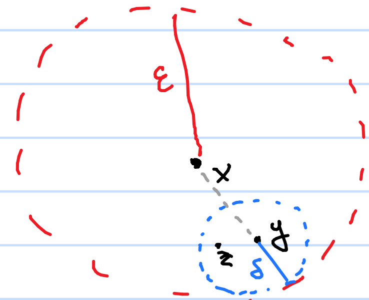
$$\delta := \varepsilon - d(x,y) > 0.$$

We claim $B_d(y,\delta) \subset B_d(x,\varepsilon)$. Indeed, for $z \in B_d(y,\delta)$, the triangle inequality yields

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \delta = d(x,y) + (\varepsilon - d(x,y)) = \varepsilon.$$

Thus $d(x,z) < \delta$ and so $z \in B_d(x,\varepsilon)$. This implies $B_d(y,\delta) \subset B_\delta(x,\varepsilon)$. $\square$

Picture "proof" for Lemma:



- Fix a set X with a metric d. We will now use the previous lemma to show show that

$$\mathcal{B} := \{ B_d(x,\varepsilon) \mid x \in X,\ \varepsilon > 0 \}$$

is a basis for a topology on X. Indeed, $x \in B_d(x,1) \in \mathcal{B}$ for all $x \in X$. Also if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then the previous lemma implies there are $\delta_1, \delta_2 > 0$ so that

$$B_d(x,\delta_1) \subset B_1 \quad \text{and} \quad B_d(x,\delta_2) \subset B_2$$

Let $\delta := \min\{\delta_1, \delta_2\}$, then for $B_3 := B_d(x,\delta) \in \mathcal{B}$ we have

$$x \in B_3 \subset B_d(x,\delta_1) \cap B_d(x,\delta_2) \subset B_1 \cap B_2$$

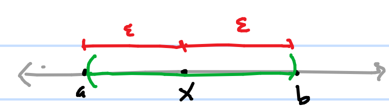
Hence $\mathcal{B}$ is a basis.

Def Let X be a set with metric d. The <u>metric topology</u> (induced by d) is the topology on X generated by the basis $\{ B_d(x,\varepsilon) \mid x \in X,\ \varepsilon > 0 \}$

Rem Just as spaces X can have many topologies, they can also have many metrics and hence many metric topologies. Thus in the above definition it is important to know <u>which</u> metric d is being used to define the basis.

Ex We revisit the same three examples as before.

1. The metric topology on $\mathbb{R}$ induced by the metric $d(x,y) = |x-y|$ is exactly the standard topology since the have a common basis of open intervals:
$$(a,b) = (x-\varepsilon, x+\varepsilon)$$



for $x = \frac{a+b}{2}$, $\varepsilon = \frac{b-a}{2}$

2. The metric topology on $\mathbb{R}^2$ induced by the distance formula is the standard topology because they have a common basis consisting of interiors of circles.

3. The metric topology induced by the discrete metric d on a set X is the discrete topology. Indeed, $B_d(x, \frac{1}{2}) = \{x\}$ for all $x \in X$. Thus $\{x\}$ is open in the metric topology and therefore all sets are open. $\square$

Prop Let X be a set with metric d. Then $U \subset X$ is open in the metric topology induced by d if and only if for all $x \in U$ there exists $\varepsilon > 0$ with $B_d(x, \varepsilon) \subset U$.

Proof Suppose U is open in the metric topology. Then for any $x \in U$ there exists a basis set $B_d(y, \delta)$ with

$$x \in B_d(y, \delta) \subset U$$

By the previous lemma, $\exists \varepsilon > 0$ so that $B_d(x, \varepsilon) \subset B_d(y, \delta) \subset U$.

Conversely, for such a U we have

$$U = \bigcup_{x \in U} B_d(x, \varepsilon)$$

and so U is open in the metric topology. □

Def We say a topological space X with topology $\mathcal{T}$ is <u>metrizable</u> if there exists a metric d on X that induces a topology $\mathcal{T}_d$ satisfying $\mathcal{T}_d = \mathcal{T}$. A <u>metric space</u> is a metrizable space with a specific metric d that induces the topology on X.

- Thus a metric space is a topological space with the additional structure of a metric that "respects" the topology in the sense that the ε-balls of the metric are open in the topology on X, and all open sets are unions of ε-balls (for varying ε).

- We saw in the examples above that the standard topologies on $\mathbb{R}$ and $\mathbb{R}^2$ are metrizable and that the discrete topology is metrizable. However, not all topologies are metrizable:

Ex $X = \{a, b\}$ with the trivial topology $\mathcal{T} = \{\emptyset, X\}$ is <u>not</u> metrizable. Indeed, if d is a metric on X, then $d(a,b) > 0$ since $a \neq b$. Thus for $\varepsilon := d(a,b)$ we have $B_d(a, \varepsilon) = \{a\} \notin \mathcal{T}$. Therefore the topology induced by d is not equal to $\mathcal{T}$. Since d was an arbitrary metric on X, we see that X is not metrizable. □

- Note that the above topological space is not Hausdorff. Thus it is also an an example of the contrapositive to the next proposition.

Prop A metrizable space is Hausdorff

Proof Let X be a metrizable space, and let d be a metric inducing the topology on X. For distinct $x, y \in X$, we have $\varepsilon := \frac{1}{2} d(x,y) > 0$. Consequently, $B_d(x, \varepsilon)$ and $B_d(y, \varepsilon)$ are disjoint neighborhoods of x and y, respectively. □

- Thus, using the results from the end of §17, we know finite sets in metric spaces are closed and convergent nets have unique limits.

Rem The additional structure on a metric space (i.e. its metric) make them easier to study than general topological spaces. For example, we'll see in the next section that in a metric space it suffices to consider sequences in lieu of nets. Consequently, it is useful to have a way to know whether or not a space is metrizable. We will see one way to tell later in the course (§34). $\square$

- While the metrizability (or lack thereof) of a topological space depends only on the topology, there are metric space properties that cannot be detected by just the topology (see the next definition). Consequently it is important to always keep straight which definitions and theorems apply to all topological spaces and which apply only to metric spaces.

Def Let X be a metric space with metric d. We say a subset $A \subset X$ is <u>bounded</u> if there exists a number $M \geq 0$ so that
$$d(a,b) \leq M$$
for all $a,b \in A$. If A is bounded and non-empty, the <u>diameter</u> of A is the quantity
$$\operatorname{diam} A := \sup_{a,b \in A} d(a,b).$$

- Note that the diameter of a set clearly depends on the particular metric d. Even boundedness depends on the metric: a set that is bounded with respect to a metric d could fail to be bounded with respect to some other metric d'. Indeed, the theorem below shows that a metric space always admits a (possibly different) metric where <u>all</u> sets are bounded. Moreover, the topologies induced by the old and new metric are the same, which shows that boundedness is <u>not</u> determined by the topology. We first need a lemma.

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Lemma Let d and d' be two metrics on a set X which induce topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that
$$B_{d'}(x,\delta) \subset B_d(x,\varepsilon).$$

Proof Consider the collections of balls under each metric:
$$\mathcal{B} = \{ B_d(x,\varepsilon) \mid x \in X, \varepsilon > 0 \} \qquad \mathcal{B}' = \{ B_{d'}(x,\varepsilon) \mid x \in X, \varepsilon > 0 \}.$$

These are bases for $\mathcal{T}$ and $\mathcal{T}'$, respectively. So by the first lemma in §13, $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $B \in \mathcal{B}$ and all $x \in B$ there exists $B' \in \mathcal{B}'$ with $x \in B' \subset B$.

($\Longrightarrow$): Let $x \in X$ and $\varepsilon > 0$. Then $x \in B := B_d(x, \varepsilon)$ and so the above implies $\exists\, B' \in \mathcal{B}'$ with $x \in B' \subset B$. Since B' is some ball in the d' metric, the first lemma in this section implies there exists $\delta > 0$ so that $B_{d'}(x, \delta) \subset B' \subset B = B_d(x, \varepsilon)$.

($\Longleftarrow$): Let $B \in \mathcal{B}$ and let $x \in B$. Since B is some ball in the d metric, the first lemma in this section implies there exists $\varepsilon > 0$ so that $B_d(x, \varepsilon) \subset B$. Letting $\delta > 0$ be as in our hypotheses yields $x \in B' := B_{d'}(x, \delta) \subset B_d(x, \varepsilon) \subset B$. Hence $\mathcal{T}' \supset \mathcal{T}$. $\square$

Thm Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by

$$\bar{d}(x,y) := \min\{d(x,y), 1\}$$

for $x, y \in X$. Then $\bar{d}$ is a metric on X that induces the same topology as d.

Proof Parts ① and ② in the definition of a metric follow for $\bar{d}$ from the corresponding properties for d. Let $x, y, z \in X$. If either $d(x,y) \geq 1$ or $d(y,z) \geq 1$ then

$$\bar{d}(x,z) \leq 1 \leq \bar{d}(x,y) + \bar{d}(y,z).$$

If $d(x,y) < 1$ and $d(y,z) < 1$, then by the triangle inequality for d we have

$$\bar{d}(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$$

Thus in either case $\bar{d}$ satisfies the triangle inequality and so is a metric.

Let $x \in X$ and $\varepsilon > 0$. We claim that

$$B_{\bar{d}}(x, \min\{\varepsilon, 1\}) \subset B_d(x, \varepsilon) \subset B_{\bar{d}}(x, \varepsilon).$$

Indeed, if y is in the first set then $\bar{d}(x,y) < \min\{\varepsilon, 1\}$. In particular, $\bar{d}(x,y) < 1$ which is only possible if $\bar{d}(x,y) = d(x,y)$. Thus $d(x,y) < \min\{\varepsilon, 1\} \leq \varepsilon$, and so $y \in B_d(x, \varepsilon)$. This proves the first of the above inclusions. For the second inclusion, let $z \in B_d(x, \varepsilon)$. Then $\bar{d}(x,z) \leq d(x,z) < \varepsilon$ and so $z \in B_{\bar{d}}(x, \varepsilon)$. This implies the second inclusion.

With the claim, we can apply the previous lemma twice to obtain that the topologies induced by d and $\bar{d}$ are equal. $\square$

Def Let d be a metric on a space X. The metric on X defined by

$$\bar{d}(x,y) := \min\{d(x,y), 1\}$$

is called the <u>standard bounded metric</u> corresponding to d.

$\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}}$ are metrizable

- We now focus on showing the above two examples when equipped with the product topology are metrizable. We must first develop a few notions specific to these examples

Def The norm of $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$ is the quantity
$$\|\mathbf{x}\|:=(x_1^2+\cdots+x_n^2)^{1/2}$$

The euclidean metric d on $\mathbb{R}^n$ is defined by
$$d(\mathbf{x},\mathbf{y}):=\|\mathbf{x}-\mathbf{y}\|=[(x_1-y_1)^2+\cdots+(x_n-y_n)^2]^{1/2}$$

The square metric ρ on $\mathbb{R}^n$ is defined by
$$\rho(\mathbf{x},\mathbf{y}):=\max\{|x_1-y_1|,\dots,|x_n-y_n|\}$$

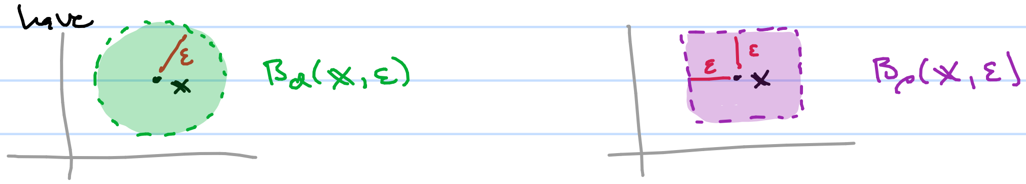
- The proof that the euclidean metric is actually a metric requires a bit of work, but we leave it to the the homework. We do, however, check that the square metric is a metric:

1: $\rho(\mathbf{x},\mathbf{y})\geq 0$ since $|x_i-y_i|\geq 0$; moreover, since $0\leq|x_i-y_i|\leq\rho(\mathbf{x},\mathbf{y})$ for each $i=1,\dots,n$, $\rho(\mathbf{x},\mathbf{y})=0$ iff $|x_i-y_i|=0$ for each i and therefore $\mathbf{x}=\mathbf{y}$.

2: $\rho(\mathbf{x},\mathbf{y})=\rho(\mathbf{y},\mathbf{x})$ follows from $|x_i-y_i|=|-(y_i-x_i)|=|y_i-x_i|$ for each $i=1,\dots,n$.

3: For each $i=1,\dots,n$ we have
$$|x_i-z_i|=|x_i-y_i+y_i-z_i|\leq|x_i-y_i|+|y_i-z_i|\leq\rho(\mathbf{x},\mathbf{y})+\rho(\mathbf{y},\mathbf{z})$$
Consequently
$$\rho(\mathbf{x},\mathbf{z})=\max_{i=1,\dots,n}|x_i-z_i|\leq\rho(\mathbf{x},\mathbf{y})+\rho(\mathbf{y},\mathbf{z})$$
So ρ is indeed a metric.

- Note that for $n=1$, d and ρ both equal the standard metric on $\mathbb{R}$: $|x-y|$. For $n=2$, we have



We have already seen that the basis consisting of such collections yield the same topology on $\mathbb{R}^2$. It turns out out that d and ρ induce the same topology on $\mathbb{R}^n$ for all $n\in\mathbb{N}$:

Thm The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are both equal to the product topology on $\mathbb{R}^n$ (where $\mathbb{R}$ has the standard topology).

Proof For $x, y \in \mathbb{R}^n$ and $i = 1, \dots, n$ we have

$$|x_i - y_i| = \sqrt{(x_i - y_i)^2} \le \underbrace{\sqrt{(x_1-y_1)^2 + \cdots + (x_n - y_n)^2}}_{= d(x,y).} \le \sqrt{n \cdot \max_{1 \le j \le n} (x_j - y_j)^2} = \sqrt{n}\, \rho(x, y)$$

Since this holds for each $i = 1, \dots, n$ we have

$$\rho(x,y) \le d(x,y) \le \sqrt{n}\, \rho(x,y)$$

Consequently for $\varepsilon > 0$ we have

$$B_\rho(x, \tfrac{1}{\sqrt{n}}\varepsilon) \subset B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$$

The previous lemma therefore implies ρ and d induce the same topology on $\mathbb{R}^n$. Thus it suffices to show the topology induced by ρ is the product topology.

Consider a product of open intervals

$$B := (a_1, b_1) \times \cdots \times (a_n, b_n)$$

Let $x = (x_1, \dots, x_n) \in B$. Set $\varepsilon_i := \min\{x_i - a_i, b_i - x_i\}$ so that

$$(x_1 - \varepsilon_1, x_1 + \varepsilon_1) \times \cdots \times (x_n - \varepsilon_n, x_n + \varepsilon_n) \subset B.$$

Letting $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_n\}$ we then have $x \in B_\rho(x, \varepsilon) \subset B$. Thus the topology induced by ρ is finer than the product topology (since sets like B form a basis). Conversely, for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$ we have

$$B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$$

which is open the product topology, which is therefore finer than the topology induced by ρ (and equivalently by d). $\square$

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- In particular, the above theorem implies the product topology on $\mathbb{R}^n$ is metrizable. Recall that the box topology equals the product topology in this case. We will now consider $\mathbb{R}^{\mathbb{N}}$ where the product and box topologies are different.

- A seemingly natural way to adapt the euclidean and square metrics are

$$\left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2} \qquad \text{and} \qquad \sup_{i \in \mathbb{N}} |x_i - y_i|$$

But these are not defined for all $x, y \in \mathbb{R}^{\mathbb{N}}$. The former makes sense iff the series converges while the latter makes sense iff the sequence is bounded. A slight tweak of the latter will give us a metric however: replace $d(x_i, y_i) = |x_i - y_i|$ with $\bar{d}(x_i, y_i) = \min\{d(x_i, y_i), 1\}$, the standard bounded metric on $\mathbb{R}$. Then $\{\bar{d}(x_i, y_i) \mid i \in \mathbb{N}\}$ always has an upper bound of 1 and hence the supremum exists. In fact, this works for any indexing set J.

Def Given an index set J and $\mathbf{x} = (x_j)_{j\in J}, \mathbf{y} = (y_j)_{j\in J} \in \mathbb{R}^J$, the equation

$$\bar{\rho}(\mathbf{x},\mathbf{y}) := \sup\{\bar{d}(x_j, y_j) \mid j \in J\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$, defines a metric on $\mathbb{R}^J$ called the <u>uniform metric</u>. The topology induced by $\bar{\rho}$ is called the <u>uniform topology</u> on $\mathbb{R}^J$.

Exercise verify that $\bar{\rho}$ is a metric.

Thm For an index set J, the uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. When J is infinite, these three topologies are all distinct.

Proof We first show the uniform topology is finer than the product topology. Let $\prod_{j\in J} U_j$ be a basis set for the product topology: $U_j \subset \mathbb{R}$ is open for all $j \in J$ but $U_j = \mathbb{R}$ for all but finitely many $j \in J$. Let $\mathbf{x} = (x_j)_{j\in J} \in \prod_{j\in J} U_j$. Let $J_0 = \{j \in J \mid U_j \neq \mathbb{R}\}$, and for each $j \in J_0$ let $\varepsilon_j > 0$ be such that

$$B_{\bar{d}}(x_j, \varepsilon_j) \subset U_j$$

Set $\varepsilon := \min\{\varepsilon_j \mid j \in J_0\}$, which is positive since $\varepsilon_j > 0$ and J_0 is finite. Then for all $j \in J$ we have $B_{\bar{d}}(x_j, \varepsilon) \subset U_j$. (For $j \in J_0$ this is because $\varepsilon \leq \varepsilon_j$, and for $j \notin J_0$ this is because $U_j = \mathbb{R}$.) Thus

$$\mathbf{x} \in B_{\bar{\rho}}(\mathbf{x}, \varepsilon) \subset \prod_{j\in J} B_{\bar{d}}(x_j, \varepsilon) \subset \prod_{j\in J} U_j$$

since if $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \varepsilon)$, then $\bar{d}(x_j, y_j) < \varepsilon$ for all $j \in J$. This shows the uniform topology is finer than the product topology.

Next, let $\mathbf{x} = (x_j)_{j\in J} \in \mathbb{R}^J$ and let $\varepsilon > 0$. We claim

$$\prod_{j\in J} (x_j - \tfrac{1}{2}\varepsilon, x_j + \tfrac{1}{2}\varepsilon) \subset B_{\bar{\rho}}(\mathbf{x}, \varepsilon).$$

Indeed, for $\mathbf{y} = (y_j)_{j\in J}$ in the set on the left we have

$$\bar{d}(x_j, y_j) \leq |x_j - y_j| < \tfrac{1}{2}\varepsilon$$

for each $j \in J$. Hence

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup_{j\in J} \bar{d}(x_j, y_j) \leq \tfrac{1}{2}\varepsilon < \varepsilon$$

so that $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \varepsilon)$. The cartesian product $\prod (x_j - \frac{1}{2}\varepsilon, x_j + \frac{1}{2}\varepsilon)$ is open in the box topology, and so this inclusion implies the box topology is finer than the uniform topology.

Finally, suppose J is infinite. Then for $\mathbf{x} \in \mathbb{R}^J$, $B_{\bar{\rho}}(\mathbf{x}, 1)$ is open in the uniform topology but <u>not</u> the product topology, while $\prod_{j\in J} (x_j - 1, x_j + 1)$ is open in the box topology but <u>not</u> the uniform topology. We leave the details as an exercise. $\square$

• The previous theorem does not rule out the box or product topologies on $\mathbb{R}^J$ being metrizable for infinite J, it just shows that if they are metrizable then $\bar{\rho}$ is not the right metric. It turns out neither topology is metrizable when J is uncountable, and the box topology is metrizable only when J is finite (so that the box and product topologies are the same). We will prove these facts in the next section. For now, we show that the product topology is metrizable when J is countable; namely, when $J = \mathbb{N}$.

Thm Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. For $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ define

$$D(\mathbf{x},\mathbf{y}) := \sup_{n\in\mathbb{N}} \frac{1}{n}\bar{d}(x_n,y_n)$$

} note that "$\frac{1}{j}$" would not have made sense for $j\in J$.

Then D is a metric that induces the product topology on $\mathbb{R}^{\mathbb{N}}$.

Proof We first show D is a metric. Clearly $D(\mathbf{x},\mathbf{y}) \geq 0$, and if $D(\mathbf{x},\mathbf{y}) = 0$ then we have for all $n\in\mathbb{N}$

$$\frac{1}{n}\bar{d}(x_n,y_n) = 0 \Rightarrow \bar{d}(x_n,y_n) = 0 \Rightarrow x_n = y_n.$$

Thus $\mathbf{x} = \mathbf{y}$. Next $D(\mathbf{x},\mathbf{y}) = D(\mathbf{y},\mathbf{x})$ is clear from the corresponding property of $\bar{d}$. Finally, we check the triangle inequality. Let $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^{\mathbb{N}}$. For each $n\in\mathbb{N}$ we have

$$\frac{1}{n}\bar{d}(x_n,z_n) \leq \frac{1}{n}\bar{d}(x_n,y_n) + \frac{1}{n}\bar{d}(y_n,z_n) \leq D(\mathbf{x},\mathbf{y}) + D(\mathbf{y},\mathbf{z})$$

Since this holds for all $n\in\mathbb{N}$, it holds for the supremum on the left:

$$D(\mathbf{x},\mathbf{z}) = \sup_{n\in\mathbb{N}} \frac{1}{n}\bar{d}(x_n,y_n) \leq D(\mathbf{x},\mathbf{y}) + D(\mathbf{y},\mathbf{z}).$$

So D is a metric as claimed.

Next, let $\mathbf{x}\in\mathbb{R}^{\mathbb{N}}$ and $\varepsilon>0$. We will find $U\subset\mathbb{R}^{\mathbb{N}}$ open in the product topology satisfying $\mathbf{x}\in U\subset B_D(\mathbf{x},\varepsilon)$, which will imply the product topology is finer than the topology induced by D. Let $N\in\mathbb{N}$ be such that $N\geq\frac{1}{\varepsilon}$, and denote

$$U := (x_1-\varepsilon, x_1+\varepsilon)\times\cdots\times(x_N-\varepsilon, x_N+\varepsilon)\times\mathbb{R}\times\mathbb{R}\times\cdots\subset\mathbb{R}^{\mathbb{N}}.$$

If $\mathbf{y}\in U$ then for $n>N$ we have

$$\frac{1}{n}\bar{d}(x_n,y_n) \leq \frac{1}{n}\cdot 1 < \frac{1}{N}$$

Thus

$$D(\mathbf{x},\mathbf{y}) \leq \max\left\{\frac{1}{1}\bar{d}(x_1,x_1),\ldots,\frac{1}{N}\bar{d}(x_N,x_N),\frac{1}{N}\right\} < \max\left\{\frac{1}{1}\varepsilon,\ldots,\frac{1}{N}\varepsilon,\frac{1}{N}\right\} \leq \varepsilon.$$

Hence $\mathbf{y}\in B_D(\mathbf{x},\varepsilon)$ and so $U\subset B_D(\mathbf{x},\varepsilon)$ as needed.

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Finally, toward showing the product topology is also coarser than the topology generated by D, let $\prod_{n\in\mathbb{N}} U_n$ be a basis set for the product topology and let $\mathbf{x}$ be an element it contains. Then $F := \{n\in\mathbb{N} \mid U_n\neq\mathbb{R}\}$ is finite and for each $n\in F$ we can find $0<\varepsilon_n<1$ so that $(x_n-\varepsilon_n, x_n+\varepsilon_n)\subset U_n$.

Set

$$\varepsilon := \min_{n\in F} \frac{\varepsilon_n}{n}$$

which is positive since $\varepsilon_n > 0$ and F is finite. We claim $B_D(x,\varepsilon) \subset \prod_{n\in\mathbb{N}} U_n$. Indeed, for $y \in B_D(x,\varepsilon)$ we have for $n\in F$ that

$$\frac{1}{n}\bar{d}(x_n,y_n) \leq D(x,y) < \varepsilon \leq \frac{\varepsilon_n}{n}$$

Hence $\bar{d}(x_n,y_n) < \varepsilon_n$. Since $\varepsilon_n < 1$, we must have $\bar{d}(x_n,y_n) = |x_n - y_n|$ and so $y_n \in (x_n - \varepsilon_n, x_n + \varepsilon_n) \subset U_n$. For $n \notin F$, we have $U_n = \mathbb{R}$ and so $y_n \in U_n$. Thus $y \in \prod_{n\in\mathbb{N}} U_n$ and so $B_D(x,\varepsilon) \subset \prod_{n\in\mathbb{N}} U_n$. This implies the topology induced by D is finer than the product topology. $\square$

§21 The Metric Topology (part 2)

- In these section we study how metrizability relates to continuity of functions. Since a metric gives us a precise notion of distance, the ε-δ definition of continuity makes sense in this context. The first theorem shows it is equivalent to the definition we have already been using.

Thm Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is continuous at $x \in X$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ so that for $x' \in X$

$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$

Consequently, f is continuous if and only if this holds for all $x \in X$.

Proof ($\Rightarrow$): Suppose f is continuous at x. Let $\varepsilon > 0$. Since $B_{d_Y}(f(x), \varepsilon)$ is a neighborhood of $f(x)$, there exists a neighborhood U of x satisfying $f(U) \subset B_{d_Y}(f(x), \varepsilon))$. By a proposition from last section, there exists $\delta > 0$ such that $B_{d_X}(x, \delta) \subset U$. Thus if $d_X(x, x') < \delta$ then $x' \in B_{d_X}(x, \delta) \subset U$, which implies $f(x') \in f(U) \subset B_{d_Y}(f(x), \varepsilon)$, and so $d_Y(f(x), f(x')) < \varepsilon$. That is, $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$.

($\Leftarrow$): Let $V \subset Y$ be a neighborhood of $f(x)$. We must find a neighborhood U of x such that $f(U) \subset V$. By a proposition from last section, there exists $\varepsilon > 0$ so that $B_{d_Y}(f(x), \varepsilon) \subset V$. Our hypotheses implies there exists $\delta > 0$ so that for $x' \in X$, $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$. Choose $U := B_{d_X}(x, \delta)$. Then for $x' \in U$, the above implies $f(x') \in B_{d_Y}(f(x), \varepsilon) \subset V$. Hence $f(U) \subset V$.

The final part of the theorem follows from continuity being equivalent to continuity at x for all $x \in X$. $\square$

- Another nice feature of metrizable spaces is that nets can often be replaced with sequences.

Lemma (The Sequence Lemma)

Let X be metrizable. For a subset $A \subset X$, $x \in \bar{A}$ if and only if there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset A$ converging to A.

Proof ($\Rightarrow$) Suppose $x \in \bar{A}$. Let d be a metric on X that induces the topology on X. Recall that we previously showed $x \in \bar{A}$ iff every neighborhood of x intersects A. Consequently, for each $n \in \mathbb{N}$, we can choose $x_n \in B_d(x, \frac{1}{n}) \cap A$. We claim that $(x_n)_{n \in \mathbb{N}}$ converges to x. Indeed, if U is a neighborhood of x, then by a proposition from last section there exists $\delta>0$ such that $B_d(x,\delta)\subset U$. Let $n_0\in\mathbb{N}$ be such that $n_0>\frac{1}{\delta}$. Then for all $n\geq n_0$ we have $\frac{1}{n}<\delta$, and so

$$x_n\in B_d(x,\tfrac{1}{n})\subset B_d(x,\delta)\subset U.$$

Thus $x_n\in U$ for all $n\geq n_0$, and then $(x_n)_{n\in\mathbb{N}}$ converges to x.

($\Leftarrow$): Suppose $(x_n)_{n\in\mathbb{N}}\subset A$ converges to x. Since $(x_n)_{n\in\mathbb{N}}\subset\bar{A}$ and a sequence is in particular a net, we know from a theorem in §17 that $x\in\bar{A}$. □

Rem There is a more abstract way to prove ($\Rightarrow$) in the above lemma. We say a topological space X has a countable basis at $x\in X$ if there exists a countable collection $\{U_n\mid n\in\mathbb{N}\}$ of open neighborhoods of x such that any other neighborhood U of x contains U_n for at least one $n\in\mathbb{N}$. We say X is first countable if it has a countable basis at x for all $x\in X$. Metric spaces are first countable: $\{B_d(x,\frac{1}{n})\mid n\in\mathbb{N}\}$ is a countable basis at x. This was all we really used in the above proof, so if one only assumes X is first countable then the same proof works by replacing $B_d(x,\frac{1}{n})$ with

$$U_1\cap U_2\cap\cdots\cap U_n$$

where $\{U_n\mid n\in\mathbb{N}\}$ is a countable basis at x. Consequently, the next theorem holds if X and Y are only assumed to be first countable.

Thm If X and Y are metrizable, then a function $f:X\to Y$ is continuous if and only if whenever a sequence $(x_n)_{n\in\mathbb{N}}\subset X$ converges to $x\in X$ one then has that the sequence $(f(x_n))_{n\in\mathbb{N}}\subset Y$ converges to $f(x)\in Y$.

Proof ($\Rightarrow$): We already proved this for nets in §18, and sequences are also nets.

($\Leftarrow$): Recall that it suffices, by a theorem from §18, to show for $A\subset X$ that $f(\bar{A})\subset\overline{f(A)}$. Let $x\in\bar{A}$. By the sequence lemma there exists $(x_n)_{n\in\mathbb{N}}\subset A$ converging to x. Thus our hypotheses imply $(f(x_n))_{n\in\mathbb{N}}$, which is contained in $f(A)$, converges to $f(x)$. So the sequence lemma then implies $f(x)\in\overline{f(A)}$. Hence $f(\bar{A})\subset\overline{f(A)}$. □

EX We use the sequence lemma to show certain product spaces are not metrizable

1. $\mathbb{R}^{\mathbb{N}}$ with the box topology is not metrizable. Consider

$$A:=\{(x_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}\mid x_n>0 \text{ for all } n\in\mathbb{N}\}$$

Denote by $\mathbf{0}$ the sequence of all zeros. Then $\mathbf{0}\in\bar{A}$. Indeed, if U is a neighborhood of $\mathbf{0}$, then there exists open intervals $(a_n,b_n)\subset\mathbb{R}$, $n\in\mathbb{N}$ such that

$$\mathbf{0}\in\prod_{n\in\mathbb{N}}(a_n,b_n)\subset U.$$

Consequently, $b_n>0$ and so

$$(\tfrac{1}{2}b_n)_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}(a_n,b_n)\cap A\subset U\cap A.$$

Thus $U\cap A\neq\emptyset$, and $\mathbf{0}\in\bar{A}$. However, no sequence in A converges to $\mathbf{0}$. Indeed, Let $(\mathbf{x}^{(k)})_{k\in\mathbb{N}} \subset A$; that is, $\mathbf{x}^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}}$ and $x_n^{(k)} > 0$ for all $n \in \mathbb{N}$. Define

$$U := \prod_{n\in\mathbb{N}} (-x_n^{(n)}, x_n^{(n)})$$

Then U is a neighborhood of $\mathbf{0}$, but $\mathbf{x}^{(k)} \notin U$ for any $k \in \mathbb{N}$ since $x_k^{(k)} \notin (-x_k^{(k)}, x_k^{(k)})$. Thus $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$ cannot converge to $\mathbf{0}$.
If $\mathbb{R}^{\mathbb{N}}$ with the box topology were metrizable, this would contradict the sequence lemma.

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2. Let J be an uncountable set. $\mathbb{R}^J$ with the product topology is not metrizable. Consider

$$A := \{(x_j)_{j\in J} \in \mathbb{R}^J \mid x_j = 1 \text{ for all but finitely many } j \in J\}.$$

Let $\mathbf{0}$ be the J-tuple whose coordinates are all zero. Then $\mathbf{0} \in \overline{A}$. Indeed, if U is a neighborhood of $\mathbf{0}$, then there exists a cartesian product satisfying

$$\mathbf{0} \in \prod_{j\in J} U_j \subset U$$

where $U_j \subset \mathbb{R}$ is open for all $j \in J$ and $U_j = \mathbb{R}$ for all but finitely many $j \in J$. Let $J_0 = \{j \in J \mid U_j \neq \mathbb{R}\}$, and define

$$x_j = \begin{cases} 0 & \text{if } j \in J_0 \\ 1 & \text{if } j \notin J_0. \end{cases}$$

Then $x_j \in U_j$ for all $j \in J$, so $(x_j)_{j\in J} \in \prod U_j \subset U$. Since J_0 is finite, we also have $(x_j)_{j\in J} \in A$. Thus $U \cap A \neq \emptyset$ and so $\mathbf{0} \in \overline{A}$. However, no sequence $(\mathbf{x}^{(n)})_{n\in\mathbb{N}} \subset A$ converges to $\mathbf{0}$. Indeed, let $J_n := \{j \in J \mid x_j^{(n)} \neq 1\}$. Then J_n is finite for all $n \in \mathbb{N}$ and consequently $\bigcup_{n\in\mathbb{N}} J_n$ is countable. Since J is uncountable, $\exists\, i \in J \setminus \bigcup_{n\in\mathbb{N}} J_n$. By definition of J_n, we have $x_i^{(n)} = 1$ for all $n \in \mathbb{N}$. But then

$$\mathbf{0} \in \pi_i^{-1}((-1,1)) \not\ni \mathbf{x}^{(n)}$$

for all $n \in \mathbb{N}$. Thus $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ cannot converge to $\mathbf{0}$. If $\mathbb{R}^J$ were metrizable, this would contradict the sequence lemma. $\square$

$\boxed{\text{Lemma}}$ Let $\mathbb{R}$ have the standard topology. The operations of addition, subtraction, and multiplication are continuous when viewed as functions $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, where $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$ has the product topology. Division is continuous when viewed as a function $\mathbb{R}\times(\mathbb{R}\setminus\{0\}) \to \mathbb{R}$, where $\mathbb{R}\setminus\{0\}$ has the subspace topology and $\mathbb{R}\times(\mathbb{R}\setminus\{0\})$ has the product topology.

- You proved the continuity of the first three on Homeworks 5 and 6. Viewing all of the above spaces as metric spaces makes the proofs even simpler. The continuity of division is left as an exercise.

Thm Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous functions where $\mathbb{R}$ has the standard topology. Then the functions $f+g,\ f-g,\ f \cdot g : X \to \mathbb{R}$ are continuous. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbb{R}$ is continuous.

Proof Consider $h : X \to \mathbb{R} \times \mathbb{R}$ defined by

$$h(x) = (f(x), g(x)).$$

Then h is continuous by the last theorem in §18. Then $f+g,\ f-g,\ f g,\ f/g$ are compositions of this function with the operations $+, -, \times, \div$, respectively, which are continuous by the previous lemma. $\square$

Def Let X be a set and let Y be a metric space with metric d. For each $n \in \mathbb{N}$, let $f_n : X \to Y$ be a function. We say the sequence of functions $(f_n)_{n \in \mathbb{N}}$ converges pointwise to a function $f : X \to Y$ if for all $x \in X$, the sequence $(f_n(x))_{n \in \mathbb{N}} \subset Y$ converges to $f(x)$.

We say $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that

$$d(f_n(x), f(x)) < \varepsilon$$

for all $n \geq N$ and all $x \in X$.

- Uniform convergence implies pointwise convergence: given $x \in X$ let $V \subset Y$ be a neighborhood of $f(x)$. Then there exists $\varepsilon > 0$ so that $B_d(f(x), \varepsilon) \subset V$. Letting $n_0 \in \mathbb{N}$ be as in the definition of uniform convergence for ε, we have $f_n(x) \in B_d(f(x), \varepsilon) \subset V$ for all $n \geq n_0$. Hence $(f_n(x))_{n \in \mathbb{N}}$ converges to $f(x)$. The converse is not true, as the following example demonstrates.

EX Let $X = [0,1]$ and let $Y = \mathbb{R}$ equipped with the standard metric $d(x,y) = |x-y|$. Consider $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Then $(f_n)_{n \in \mathbb{N}}$ converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

For $x=1$, $f_n(x) = 1$ for all $n \in \mathbb{N}$ this is clear. For $0 \leq x < 1$, let $0 < \varepsilon < 1$ and choose $n_0 \in \mathbb{N}$ so that $n_0 > \frac{\log(\varepsilon)}{\log(x)}$. Then for $n \geq n_0$ we have

$$n > \frac{\log(\varepsilon)}{\log(x)} \Rightarrow n \log(x) < \log(\varepsilon) \Rightarrow \log(x^n) < \log(\varepsilon) \Rightarrow x^n < \varepsilon.$$

Since $x^n \geq 0$, we have $|x^n - 0| = x^n < \varepsilon$ for all $n \geq n_0$, and so $(f_n(x))_{n \in \mathbb{N}}$ converges to 0.

However $(f_n)_{n \in \mathbb{N}}$ does not converge uniformly to f. Let $\varepsilon = \frac{1}{2}$. Then for any $n \in \mathbb{N}$ there exists $x \in [0,1)$ with $|f_n(x) - f(x)| \geq \varepsilon$. Indeed, let $x \in [\varepsilon^{1/n}, 1)$ so that $\varepsilon \leq x^n < 1$, then $|f_n(x) - f(x)| = |x^n - 0| = x^n \geq \varepsilon$. $\square$

- Observe that f in the previous example is <u>not</u> continuous: the net $(t)_{t\in[0,1)}$ where $[0,1)$ is given its usual order converges to 1, but $(f(t))_{t\in[0,1)} = (0)_{t\in[0,1)}$ converges to $0 \neq 1 = f(1)$. On the other hand, each f_n was continuous by Homework 6. Thus continuous functions can converge pointwise to functions that are not continuous. The next theorem shows this is <u>not</u> the case for uniform convergence. (This also yields an alternate proof that the sequence in the previous example does not converge uniformly)

Thm (Uniform Limit Theorem)
Let X be a topological space and let Y be a metric space with metric d. For each $n\in\mathbb{N}$, let $f_n: X\to Y$ be a continuous function. If $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f: X\to Y$, then f is continuous.

Proof We will show f is continuous at every $x\in X$. Fix $x_0\in X$ and let $V\subset Y$ be a neighborhood of $f(x_0)$. Then there exists $\varepsilon>0$ so that $B_d(f(x_0),\varepsilon)\subset V$. Using uniform convergence, let $n_0\in\mathbb{N}$ be such that

$$d(f_n(x), f(x)) < \frac{\varepsilon}{3}$$

for all $n\geq n_0$ and all $x\in X$. Using the continuity of f_{n_0} at x_0, there exists a neighborhood U of x_0 satisfying $f_{n_0}(U)\subset B_d(f_{n_0}(x_0), \frac{\varepsilon}{3})$. We claim $f(U)\subset V$. Indeed, if $x\in U$, then

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f(x_0)) \\ &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(x_0)) + d(f_{n_0}(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus $f(x)\in B_d(f(x_0),\varepsilon)\subset V$ and so $f(U)\subset V$. This shows f is continuous at x_0, and since $x_0\in X$ was arbitrary we see that f is continuous. $\square$

- Recall that Y^X denotes the set of functions from X to Y, which can also be regarded as X-tuples. Consequently we can define the uniform metric $\bar{\rho}$ on this space:

$$\bar{\rho}(f,g) := \sup_{x\in X} \bar{d}(f(x), g(x))$$

A sequence $(f_n)_{n\in\mathbb{N}}\subset Y^X$ converges to $f\in Y^X$ in the topology induced by this metric if and only if it converges uniformly to f (<u>Exercise</u>: verify!)

§ 22 The Quotient Topology

- Let X be a topological space and let Y be a set (with no topology). Suppose there is a surjective function $p: X \to Y$. We can use this to define a topology on Y:

$$\mathcal{T} := \{ U \subset Y \mid p^{-1}(U) \subset X \text{ is open} \}.$$

Let's check this is indeed a topology:

1: $p^{-1}(\phi) = \phi$ and $p^{-1}(Y) = X$ are both open in X, so $\phi, Y \in \mathcal{T}$.

2: If $S \subset \mathcal{T}$ is a subcollection then

$$p^{-1}\Big(\bigcup_{U \in S} U \Big) = \bigcup_{U \in S} p^{-1}(U)$$

is open in X. So $\mathcal{T}$ is closed under arbitrary unions.

3: If $U_1, \ldots, U_n \in \mathcal{T}$, then

$$p^{-1}(U_1 \cap \cdots \cap U_n) = p^{-1}(U_1) \cap \cdots \cap p^{-1}(U_n)$$

is open in X. So $\mathcal{T}$ is closed under finite intersections.

Def Given a surjective function $p: X \to Y$ from a topological space X to a set Y, the quotient topology on Y induced by p is

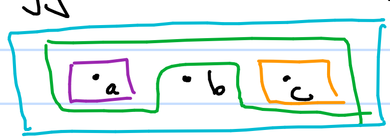
$$\mathcal{T} := \{ U \subset Y \mid p^{-1}(U) \subset X \text{ is open} \}.$$

Ex 1 Let $X = \mathbb{R}$ and $Y = \{a, b, c\}$. Define $p: X \to Y$ by

$$p(t) = \begin{cases} a & \text{if } t < 0 \\ b & \text{if } t = 0 \\ c & \text{if } t > 0 \end{cases}$$

Observe that $p^{-1}(\{a\}) = (-\infty, 0)$ and $p^{-1}(\{c\}) = (0, \infty)$ are open, so $\{a\}$ and $\{c\}$ are open in the quotient topology on Y induced by p. However $p^{-1}(\{b\}) = \{0\}$ is not open and so $\{b\}$ is not open in the quotient topology. The quotient topology on Y consists of the following sets:

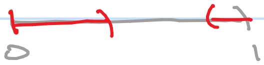
Y:



2 Let $X = [0,1]$ with the subspace topology from the standard topology on $\mathbb{R}$. Let $Y = [0,1)$ and define $p: X \to Y$ by

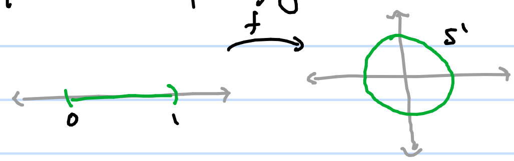
$$p(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t = 1 \end{cases}$$

Then $[0, \frac{1}{2})$ is not open in the quotient topology on Y induced by p. Indeed $p^{-1}([0, \frac{1}{2})) = [0, \frac{1}{2}) \cup \{1\}$, which is not open in X. Sets of the following types are open in the quotient topology:




In fact such sets form a basis for the quotient topology. Moreover, recall that

$$f: [0,1) \to S^1$$
$$f(t) = (\cos(2\pi t), \sin(2\pi t))$$



is a bijection, but not a homeomorphism when $[0,1)$ is given the subspace topology. If we instead give $[0,1)$ the quotient topology then this <u>is</u> a homeomorphism. We'll see this at the end of the section $\square$

- Note that $p: X \to Y$ is continuous when Y is given the quotient topology: if $V \subset Y$ is open then by definition of the quotient topology it must be that $p^{-1}(V) \subset X$ is open. That is, $V \subset Y$ being open implies $p^{-1}(V) \subset X$ is open. We actually having a stronger condition than continuity in this case because $V \subset Y$ is open if and only if $p^{-1}(V) \subset X$ is open.

Def Let X and Y be topological spaces and let $p: X \to Y$ be a surjective map. We say p is a <u>quotient map</u> if a subset $V \subset Y$ is open if and only if $p^{-1}(V) \subset X$ is open.

- Note that when $p: X \to Y$ is a quotient map, the topology on Y is necessarily equal to the quotient topology induced by p.

- One could equivalently define quotient maps by requiring $B \subset Y$ be closed if and only if $p^{-1}(B) \subset X$ is closed since
$$p^{-1}(Y \setminus B) = X \setminus p^{-1}(B).$$

Ex Let X and Y be topological spaces. We say $f: X \to Y$ is an <u>open map</u> if $f(U) \subset Y$ is open for all $U \subset X$ open. If f is surjective, open, and continuous, then f is a quotient map. Indeed, if $V \subset Y$ is open, then $f^{-1}(V)$ is open by continuity of f. If $f^{-1}(V) \subset X$ is open, then
$$V = f(f^{-1}(V))$$
by surjectivity, and this is open since f is an open map.
We say $f: X \to Y$ is a <u>closed map</u> if $f(A) \subset Y$ is closed for all closed $A \subset X$. A similar prove to the above shows surjective, closed, continuous maps are quotient maps. $\square$

- We note that there are maps that are closed and not open, and maps that are open but not closed. There are also quotient maps that are neither open nor closed. Examples of each can be found in the textbook.

Def Let $p: X \to Y$ be a surjective function. We say a subset $A \subset X$ is <u>saturated</u> with respect to p if

$$p^{-1}(p(A)) = A.$$

- Exercise Show $A \subset X$ is saturated iff $X \setminus A$ is saturated.

- Note that if p is injective, then every subset is saturated by Exercise 1 on Homework 1. Consequently the following proposition implies an injective quotient map is a homeomorphism.

Prop Let X and Y be topological spaces, and let $p: X \to Y$ be surjective and continuous. Then the following are equivalent

1. p is a quotient map
2. $p(U) \subset Y$ is open for all saturated open sets $U \subset X$.
3. $p(A) \subset Y$ is closed for all saturated closed sets $A \subset X$.

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Proof The equivalence of 2 and 3 follows from the above exercise and the surjectivity of p. We leave the details as an exercise.

1 $\Rightarrow$ 2 : Let $U \subset X$ be a saturated open set and denote $V := p(U)$. Thus $p^{-1}(V) = p^{-1}(p(U)) = U$, which is open and thus V is open by definition of a quotient map.

2 $\Rightarrow$ 1 : Let $V \subset Y$. If V is open, then $p^{-1}(V)$ is open by the continuity of p. Conversely, note that $U := p^{-1}(V)$ is saturated by the surjectivity of p:

$$p^{-1}(p(U)) = p^{-1}(\underbrace{p(p^{-1}(V))}_{=V}) = p^{-1}(V) = U$$

Thus $U = p^{-1}(V)$ is always saturated and if it is open then 2 implies $p(U) = p(p^{-1}(V)) = V$ is open in Y. Thus V is open in Y iff $p^{-1}(V)$ is open in X. That is, p is a quotient map. $\square$

- Let $p: X \to Y$ be a surjective function. We can define a relation on X as follows: for $x, x' \in X$ write $x \sim x'$ iff $p(x) = p(x')$. This is an equivalence relation (Exercise verify this). For $x \in X$, denote the equivalence class of x by

$$[x] = \{x' \in X : x' \sim x\},$$

and note that for $x, x' \in X$

$$[x]\cap[x'] = \begin{cases} [x] & \text{if } x\sim x' \\ \phi & \text{otherwise} \end{cases}$$

Thus if $X/\sim$ denotes the collection of equivalence classes, then it yields a partition of X:

$$X = \bigcup_{[x]\in X/\sim} [x] \quad \text{and} \quad [x]\cap[x'] = \phi \quad \forall [x],[x']\in X/\sim \text{ distinct.}$$

Also note that $X/\sim \ni [x] \mapsto p(x)\in Y$ is a bijection. Indeed, it is well-defined since $p(x') = p(x)$ for all $x'\in[x]$, and it is surjective since p is surjective. It is injective because $[x]\neq[x']$ demands $p(x)\neq p(x')$.

This allows us to identify Y with $X/\sim$ (as sets), and p induces a new map $q: X\to X/\sim$ defined by

$$q(x) = [x]$$

Recall that $\sim$ was determined by p. Conversely, given any equivalence relation $\sim$ on X, one can define the map q above for this equivalence relation and will obtain a surjective map.

$\boxed{\text{Def}}$ Let X be a topological space and let $\sim$ be an equivalence relation on X. Define $q: X\to X/\sim$ by

$$q(x) := [x]$$

We call $X/\sim$ equipped with the quotient topology induced by q a <u>quotient space</u> of X.

- For $x\in X$, observe that $\{[x]\}$ is a subset in $X/\sim$ and

$$\begin{aligned} q^{-1}(\{[x]\}) &= \{y\in X \mid q(y)\in\{[x]\}\} \\ &= \{y\in X \mid q(y) = [x]\} \\ &= \{y\in X \mid [y]=[x]\} = [x]. \end{aligned}$$

Thus $V\subset X/\sim$ is open in the quotient topology iff

$$q^{-1}(V) = q^{-1}\Big(\bigcup_{[x]\in V}\{[x]\}\Big) = \bigcup_{[x]\in V} q^{-1}(\{[x]\}) = \bigcup_{[x]\in V}[x]$$

is open. That is a subset of $X/\sim$, which is a collection of equivalence classes, will be open in $X/\sim$ iff the union of these equivalence classes is open in X.

$\boxed{\text{EX}}$ 1 Let $X = [0,1]$ and define $p: X\to[0,1)$ by

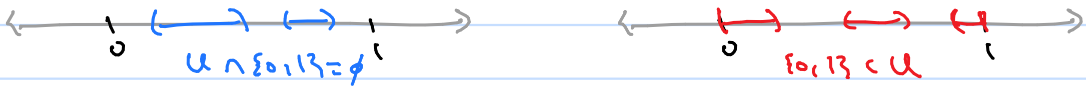
$$p(t) = \begin{cases} t & \text{if } 0\le t<1 \\ 0 & \text{otherwise.} \end{cases}$$

The induced equivalence relation is $1\sim 0$ and for $0<t<1$, $t\sim s$ iff $t=s$.

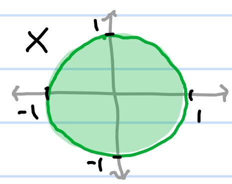
Thus if $q : X \to X/\sim$ and $A \subset X$, then

$$q^{-1}(q(A)) = \begin{cases} A \cup \{0,1\} & \text{if } A \cap \{0,1\} \neq \phi \\ A & \text{otherwise} \end{cases}.$$

This implies a set A is saturated with respect to q iff $A \cap \{0,1\} = \phi$ or if $A \cap \{0,1\} \neq \phi$ then $\{0,1\} \subset A$. Thus the open sets in the quotient space $X/\sim$ are $q(U)$ where $U \subset X$ is open and either $U \cap \{0,1\} = \phi$ or $\{0,1\} \subset U$.



② Let $X := \{ (x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \}$ equipped with the subspace topology from $X \subset \mathbb{R}^2$. Let

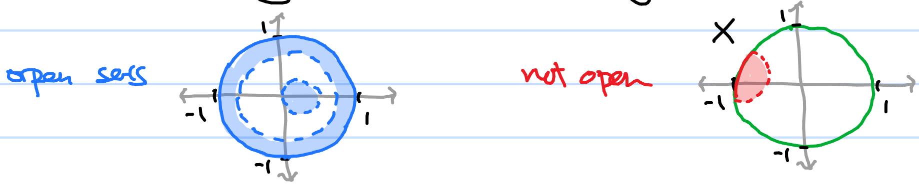


$$Y := \{ (x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1 \} \cup \{(1,0)\}$$

and define $p : X \to Y$ by

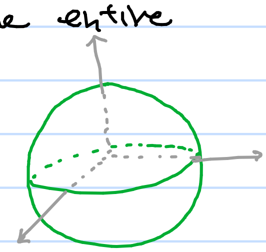
$$p(x_1,x_2) = \begin{cases} (x_1,x_2) & \text{if } x_1^2 + x_2^2 < 1 \\ (1,0) & \text{otherwise} \end{cases}$$

Define an equivalence relation on X by: $(x_1,x_2) \sim (y_1,y_2)$ iff $p(x_1,x_2) = p(y_1,y_2)$. We can invision $X/\sim$ as a copy of X but where the boundary circle is a single point.



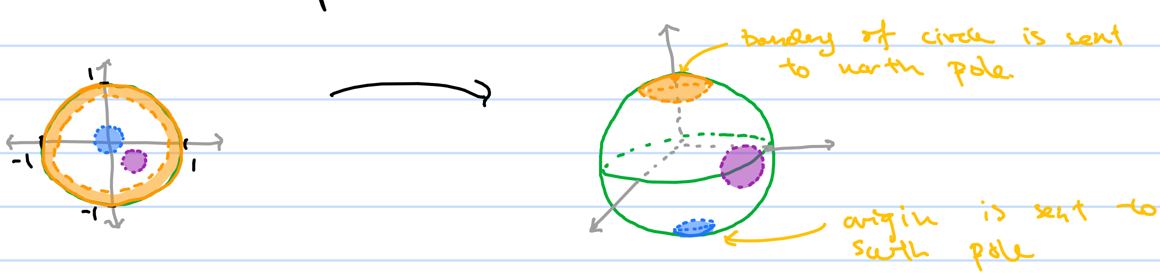
An open subset $X/\sim$ must either contain either the entire boundary circle or be disjoint from it.

Consider



$$S^2 := \{ (x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \}$$

with the subspace topology from $S^2 \subset \mathbb{R}^3$. Then $X/\sim$ and S^3 are homeomorphic:



③ Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ which we give the subspace topology. Define an equivalence relation on X by:

$$(x_1,x_2)\sim(y_1,y_2) \quad \text{iff} \quad \begin{cases} (x_1,x_2)=(y_1,y_2) & \text{or} \\ x_1,y_1\in\{0,1\} \text{ and } x_2=y_2 & \text{or} \\ x_2,y_2\in\{0,1\} \text{ and } x_1=y_1 & \text{or} \\ x_1,x_2,y_1,y_2\in\{0,1\} & \end{cases}$$

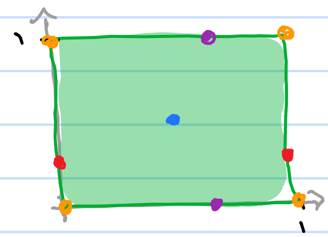
So, for example,

$[(1/2,1/2)] = \{(1/2,1/2)\}$

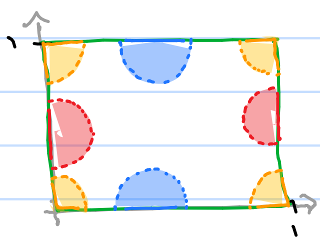
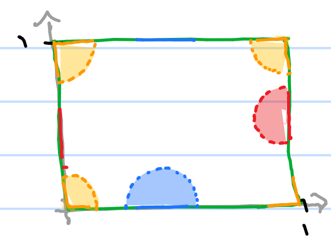
$[(0,1/3)] = \{(0,1/3),(1,1/3)\}$

$[(2/3,1)] = \{(2/3,1),(2/3,0)\}$

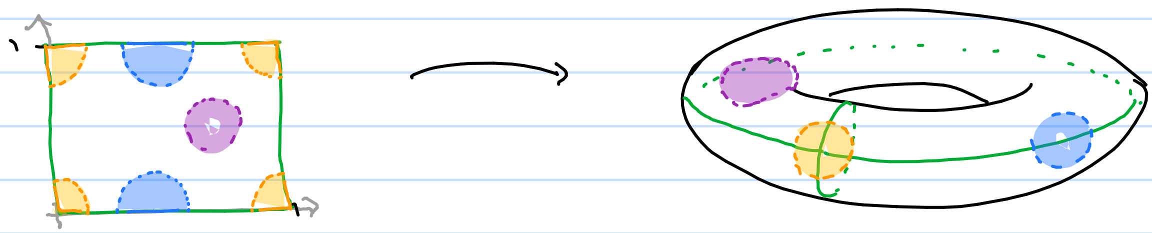
$[(0,0)] = \{(0,0),(1,0),(0,1),(1,1)\}$



Thus we can visualize $X/\sim$ as a copy of X where corresponding points on the top and bottom are identified, corresponding points on the left and right are identified, and the four corners are identified.

open sets:  not open: 

$X/\sim$ with the quotient topology is homeomorphic to the 2-torus in $\mathbb{R}^3$, which is the surface of a "donut":



$\square$

- You should visualize forming $X/\sim$ as taking a physical copy of X and gluing together any $x,y\in X$ satisfying $x\sim y$.

- We know explore the interaction between quotient topologies/maps/spaces and the previous concepts from the course.

$\boxed{\text{Thm}}$ Let $p: X\to Y$ be a quotient map and let $A\subset X$ be a subspace that is saturated with respect to p. Let $q := p|_A$, where we also restrict the range: $q: A\to p(A)$. If A is either open or closed in X, then q is a quotient map.

Proof We first claim:

$$q^{-1}(V) = p^{-1}(V) \quad \forall V\subset p(A) \quad \ast$$

Indeed, if $x \in q^{-1}(V)$ then $x \in A$ and $p(x) = q(x) \in V$. Hence $x \in p^{-1}(V)$ and so $q^{-1}(V) \subset p^{-1}(V)$. Conversely, if $x \in p^{-1}(V)$, then since $V \subset p(A)$ and A is saturated, we have

$$p^{-1}(V) \subset p^{-1}(p(A)) = A.$$

Thus $x \in A$ and so $q(x) = p(x) \in V$. Hence $x \in q^{-1}(V)$ and therefor $p^{-1}(V) \subset q^{-1}(V)$. This proves $(*)$.

Now, assume A is open (closed). For $V \subset p(A)$, if $q^{-1}(V)$ is open (closed) in X then $(*)$ implies $p^{-1}(V)$ is open (closed). Since p is a quotient map, this implies V is open (closed) in Y and hence open (closed) in $p(A)$. Conversely, suppose V is open (closed) in $p(A)$. Since A is saturated, we have $p^{-1}(p(A)) = A$, and because p is a quotient map this implies $p(A)$ is open (closed). Consequently, V being open (closed) in $p(A)$ implies Y is open (closed) in Y. Therefore $p^{-1}(V)$ is open (closed) in X, but $p^{-1}(V) = q^{-1}(V)$ by $(*)$. Thus V is open (closed) in $p(A)$ iff $q^{-1}(V)$ is open (closed) in X. That is, q is a quotient map. $\square$

Prop Let $p: X \to Y$ and $q: Y \to Z$ be quotient maps. Then $q \circ p : X \to Z$ is a quotient map

Proof Note that $q \circ p$ is surjective as a composition of surjective maps. For $U \subset Z$ we have

$$(q \circ p)^{-1}(U) = p^{-1}(q^{-1}(U)).$$

Thus U is open in Z iff $q^{-1}(U)$ is open in Y iff $p^{-1}(q^{-1}(U))$ is open in X. Therefore $q \circ p$ is a quotient map. $\square$

- Recall that we determined precisely when a function whose <u>range</u> is a <u>product</u> space is continuous: iff the coordinate functions are continuous. The next theorem is the analogue of this for functions whose <u>domain</u> is a <u>quotient</u> space.

Thm Let X, Y, and Z be topological spaces and let $p: X \to Y$ be a quotient map. Then $f: Y \to Z$ is continuous if and only if $g := f \circ p : X \to Z$ is continuous. Moreover, f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g = f \circ p"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Proof Suppose f is continuous. Then $g = f \circ p$ is continuous as a composition of continuous functions. Likewise, if f is a quotient map then so is the composition $g = f \circ p$.

Now assume g is continuous. Given $V \subset Z$ open, we must show $f^{-1}(V) \subset Y$ is open. Since g is continuous, we know $g^{-1}(V) \subset X$ is open. Observe that

$$g^{-1}(V) = (f \circ p)^{-1} = p^{-1}(f^{-1}(V)).$$

Since p is a quotient map, $p^{-1}(f^{-1}(V))$ being open implies $f^{-1}(V)$ is open. Thus f is continuous.

Finally, suppose g is a quotient map. The surjectivity of $g = f \circ p$ implies f is surjective. Given $V \subset Z$, we know $f^{-1}(V) \subset Y$ is open because g is continuous and therefore so is f by the above. Conversely, if $f^{-1}(V)$ is open, then so is

$$p^{-1}(f^{-1}(V)) = (f \circ p)^{-1}(V) = g^{-1}(V).$$

Since g is a quotient map, this implies V is open. Hence V is open if and only if $f^{-1}(V)$ is open, and therefore f is a quotient map. $\square$

- Let p, f, and g be as in the above theorem. Note that for $y \in Y$, if $x, x' \in p^{-1}(\{y\})$ then

$$g(x) = f \circ p(x) = f(y) = f \circ p(x') = g(x').$$

That is, g is constant on the subsets $p^{-1}(\{y\}) \subset X$ for all $y \in Y$. In fact, if f is not already defined and $g: X \to Z$ satisfies this condition, then we can define $f: Y \to Z$ by setting $f(y) := g(x)$ for any $x \in p^{-1}(\{y\})$. Note that $f \circ p = g$ since for $x \in X$

$$f \circ p(x) = f(p(x)) = g(x)$$

because $x \in p^{-1}(p(\{x\}))$. Thus f is continuous by the previous theorem. We say f is induced by g in this case.

Ex Let $g: X \to Z$ be surjective and continuous. Define an equivalence relation $\sim$ on X by $x \sim x'$ iff $g(x) = g(x')$. Let $Y := X/\sim$ with the quotient topology induced by the map $p(x) = [x]$. For any $[x] \in Y$ we have

$$p^{-1}(\{[x]\}) = \{x' \in X \mid g(x) = g(x')\}.$$

g is clearly constant on such sets, and so there exists continuous $f: X/\sim \to Z$ $f([x]) := g(x)$. Observe that $g(x) = f([x]) = f(p(x))$, as expected. $\square$

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Cor Let $g: X \to Z$ be surjective and continuous. Let $\sim$, $p: X \to X/\sim$, and $f: X/\sim \to Z$ be as in the previous example.

1. f is a bijection and is a homeomorphism if and only if g is a quotient map.
2. If Z is Hausdorff, then so is $X/\sim$.

Proof

① Since $f \circ p = g$, the surjectivity of g implies f is surjective. Also, if $[x], [x'] \in X/\sim$ satisfy $f([x]) = f([x'])$ then

$$g(x) = f([x]) = f([x']) = g(x').$$

Thus $x \sim x'$ and therefore $[x] = [x']$. So f is also injective.

Now, suppose f is a homeomorphism. Then in particular, f is a quotient map and hence g is a quotient map by the previous theorem.

Conversely, if g is a quotient map then so is f. So f is a injective quotient map and therefore a homeomorphism by the proposition following the definition of saturated sets.

② Suppose Z is Hausdorff. Let $[x], [x'] \in X/\sim$ be distinct. Then $g(x) \neq g(x')$ and so there exists $U, V \subset Z$ disjoint neighborhoods of $g(x)$ and $g(x')$, respectively. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint sets containing $[x]$ and $[x']$, respectively. These sets are also open since p is a quotient map and

$$p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U) = g^{-1}(U)$$
$$p^{-1}(f^{-1}(V)) = (f \circ p)^{-1}(V) = g^{-1}(V).$$

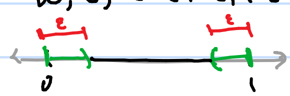
Hence $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighborhoods of $[x]$ and $[x']$, respectively. Therefore $X/\sim$ is Hausdorff. □

Ex ① For $X = [0,1]$ and $Z = S^1$ let $g : X \to Z$ be defined by

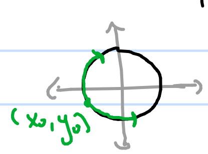
$$g(t) := (\cos(2\pi t), \sin(2\pi t)).$$

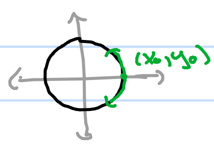
Then g is continuous since the coordinate functions $\cos(2\pi t)$ and $\sin(2\pi t)$ are continuous, and g is surjective. Furthermore, g is a quotient map. Indeed, let $U \subset X$ be a saturated open set. It suffices to show $g(U) \subset Z$ is open. Fix $(x_0, y_0) \in g(U)$. Then there exists $t_0 \in U$ such that $g(t_0) = (x_0, y_0)$. If $0 < t_0 < 1$, then there exists $\varepsilon > 0$ so that $(t_0 - \varepsilon, t_0 + \varepsilon) \subset U$. If $t_0 = 0, 1$, then $\{0, 1\} \subset U$ since U is saturated and so there exists $\varepsilon > 0$ so that $[0, \varepsilon) \cup (1-\varepsilon, 1] \subset U$.

or



These are mapped to the following respective subsets of Z:



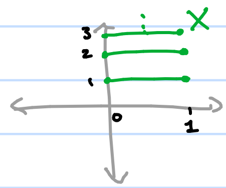


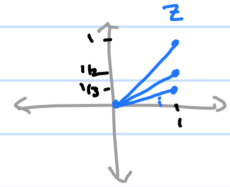
Thus (x_0, y_0) has a neighborhood contained in $g(U)$, and therefore $g(U)$ is open. Hence g is a quotient map, and the corollary implies the induced map $f : X/\sim \to Z$ is a homeomorphism.

(2) Consider the following subspaces of $\mathbb{R}^2$:

$$X := [0,1] \times \mathbb{N}$$

$$Z := \{ (t, t/n) \in \mathbb{R}^2 \mid t \in [0,1], n \in \mathbb{N} \}$$

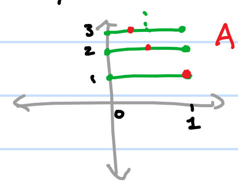




Define $g : X \to Z$ by $g(x,n) := (x, x/n)$. Then g is continuous and surjective (Exercise check this). Note that $g|_{\{0\}\times\mathbb{N}} \equiv 0$ and $g|_{X \setminus \{0\}\times\mathbb{N}}$ is injective. Thus $(x,n) \sim (x',n')$ if and only if $x = x'$ and either $n = m$ or $x = 0 = x'$.

That is $X/\sim$ looks like X but with points of the form $(0,n)$, $n \in \mathbb{N}$, indentified. Let $f : X/\sim \to Z$ be the map induced by g. Then f is a continuous bijection, but we claim it is not a homeomorphism.

The previous corollary implies it suffices to show g is not a quotient map. Consider $A := \{ (1/n, n) \mid n \in \mathbb{N} \} \subset X$. Then A is closed because it has no limit points and therefore contains all of them. A is also saturated with respect to g because $A \subset X \setminus \{0\}\times\mathbb{N}$ and g is injective on this subset. Thus if g were a quotient map then $g(A) \subset Z$ need to be closed. But



$$g(A) = \{ (\tfrac{1}{n}, \tfrac{1}{n^2}) \mid n \in \mathbb{N} \}$$

does not contain $(0,0)$ which is a limit point: $\lim_{n\to\infty} (\frac{1}{n}, \frac{1}{n^2}) = (0,0)$ since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Thus g is not a quotient map and therefore f is not a homeomorphism. $\square$

Cor If $p : X \to Y$ and $q : X \to Z$ are quotient maps such that for $x, x' \in X$ one has $p(x) = p(x')$ iff $q(x) = q(x')$, then $p(x) \mapsto q(x)$ defines a homeomorphism of Y and Z.

Proof The hypotheses imply p and q induce the same equivalence relation $\sim$ on X. Since p and q are both quotient maps, the induced maps

$$X/\sim \ni [x] \mapsto p(x) \in Y$$

$$X/\sim \ni [x] \mapsto q(x) \in Z$$

are homeomorphisms by the previous corollary. Thus the composition

$$Y \ni p(x) \mapsto [x] \mapsto q(x) \in Z$$

is a homeomorphism. $\square$