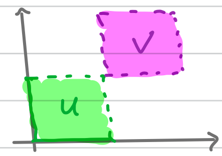
§ 23 Connected Spaces

Def Let X be a topological space. A <u>separation</u> of X is a pair of nonempty open sets $U, V \subset X$ satisfying $U \cap V = \phi$ and $U \cup V = X$. We say X is <u>disconnected</u> if a separation exists. Otherwise we say X is <u>connected</u>.

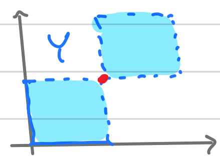
EX ① $X = [0,1) \cup (1,2) \subset \mathbb{R}$ is disconnected since $U = [0,1)$ and $V = (1,2)$ is a separation of X.

$Y = [0,2)$ is connected, but this is harder to see. We must show a separation of Y cannot exist. We'll delay proving this until the next section.

② Let $\mathbb{R}^2$ have the std. top. Then
$$X = [0,1) \times [0,1) \cup (1,2) \times (1,2)$$
is disconnected since $U = [0,1) \times [0,1)$ and $V = (1,2) \times (1,2)$ is a separation of X. (Note that U and V only need to be open in X.)



However
$$Y = X \cup \{(1,1)\}$$
is connected. We will delay proving this until later.



③ Let X be an arbitrary set. If X has the trivial topology, then X is connected. If X has the discrete topology and at least two elements, then $U = \{x\}$ and $V = X \setminus \{x\}$ is a separation of X for any $x \in X$. □

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Prop A topological space X is connected if and only if ϕ and X are the only subsets of X that are both open and closed.

Proof ($\Longrightarrow$): Assume X is connected and suppose $A \subset X$ is both open and closed. If $A = \phi$ or $A = X$ then we are done. Otherwise A is nonempty and open and $X \setminus A$ is nonempty and open. But then A and $X \setminus A$ is a separation of X, which contradicts X being connected. Thus we must have $A = \phi$ or $A = X$.

($\Longleftarrow$): We proceed by contrapositive. Assume X is disconnected. Then there exists a separation of X be nonempty open sets $U, V \subset X$. So $U \neq \phi$ and $V \neq \phi$ implies $U \neq X$. Since $U \cap V = \phi$ and $U \cup V = X$, we see that $X \setminus U = V$ and so U is closed since its complement V is open. Hence there is an open and closed subset other than ϕ and X. □

Def In a topological space X, we say $A \subset X$ is <u>clopen</u> if it is both open and closed.

• So another way to phrase the previous proposition is: "a space is connected iff the only clopen sets are ϕ and X."

Lemma Let X be a topological space and let $Y \subset X$ be a subspace. A pair of nonempty subsets $A, B \subset Y$ is a separation of Y if and only if $\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$, and $A \cup B = Y$.
equivalently no limit point of A lies in B or vice versa.

Proof ($\Longrightarrow$): Assume $A, B \subset Y$ is separation. Then A and B are closed in Y and equal their own closures in Y, which we recall are $\bar{A} \cap Y$ and $\bar{B} \cap Y$. Thus

$$\bar{A} \cap B = \bar{A} \cap Y \cap B = A \cap B = \phi$$
$$A \cap \bar{B} = A \cap Y \cap \bar{B} = A \cap B = \phi.$$

Also $A \cup B = Y$ by definition of a separation.

($\Longleftarrow$): Assume $\bar{A} \cap B = A \cap \bar{B} = \phi$ and $A \cup B = Y$. So

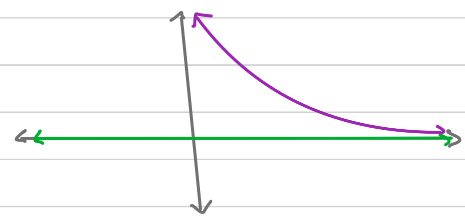
$$A \subset \bar{A} \cap Y \subset Y \setminus B \subset A$$

so that $A = \bar{A} \cap Y$ and therefore A is closed in Y. Similarly B is closed in Y. Note that $A \cap B \subset \bar{A} \cap B = \phi$ implies A and B are disjoint and $A \cup B = Y$ implies $A = Y \setminus B$ and $B = Y \setminus A$ are open. Thus A, B is a separation of Y. □

Ex ① The set

$$X = \{(x,y) \in \mathbb{R}^2 \mid y = 0\} \cup \{(x, 1/x) \in \mathbb{R}^2 \mid x > 0\}$$

is disconnected.



$$\mathcal{U} = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$$
$$V = \{(x, 1/x) \in \mathbb{R}^2 \mid x > 0\}$$

is a separation of X by the previous lemma: $\mathcal{U}, V \neq \phi$, $\mathcal{U} \cup V = X$, and

$$\bar{\mathcal{U}} \cap V = \mathcal{U} \cap V = \phi$$
$$\mathcal{U} \cap \bar{V} = \mathcal{U} \cap V = \phi.$$

② $\mathbb{Q} \subset \mathbb{R}$ is disconnected. In fact if a subspace $Y \subset \mathbb{Q}$ contains at least two distinct points p, q say with $p < q$, then there exists $a \in \mathbb{R} \setminus \mathbb{Q}$ with $p < a < q$ so that

$$\mathcal{U} := (-\infty, a) \cap \mathbb{Q} \ni p$$
$$V := (a, \infty) \cap \mathbb{Q} \ni q$$

is a separation of $\mathbb{Q}$. Thus only subspaces of the form $Y = \{a\}$ are connected. □

• We now focus on results that tell us when a set is connected. These will be put to use in the following section.

Lemma If $\mathcal{U}$ and V is a separation of X and $Y \subset X$ is a connected subspace, then either $Y \subset \mathcal{U}$ or $Y \subset V$.

Proof Since U, V are open in X

$$U \cap Y = Y \setminus (V \cap Y)$$

is clopen in Y. Since Y is connected either $U \cap Y = Y$ so that $Y \subset U$, or $U \cap Y = \phi$ so that $Y \subset V$. $\square$

Thm Let $\{Y_j \mid j \in J\}$ be a collection of connected subspaces of X. If $\bigcap_{j \in J} Y_j \neq \phi$, then $\bigcup_{j \in J} Y_j$ is connected.

Proof Let $y_0 \in \bigcap_{j \in J} Y_j$ and denote $Y := \bigcup_{j \in J} Y_j$. Suppose, towards a contradiction, that Y is disconnected with separation $U, V \subset Y$. Then $y_0 \in U$ or V, and without loss of generality we assume $y_0 \in U$. By the previous lemma, for each $j \in J$ we have either $Y_j \subset U$ or $Y_j \subset V$. But $y_0 \in U$ implies $Y_j \subset U$ for all $j \in J$. Consequently $Y = \bigcup_{j \in J} Y_j \subset U$. But this contradicts V being nonempty. Thus Y must be connected. $\square$

- The next theorem says that if you start with a connected set and add only limit points of the set, you'll still have a connected set. In particular, the closure of a connected set is connected.

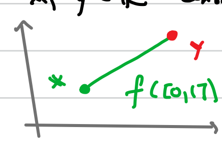
Thm Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is connected.

Proof Suppose, towards a contradiction, that B is disconnected with separation $U, V \subset B$. By the above lemma, either $A \subset U$ or $A \subset V$. Without loss of generality $A \subset U$. Then $\bar{A} \subset \bar{U}$, and by an earlier lemma $\bar{U} \cap V = \phi$. Thus $B \subset \bar{A} \subset \bar{U} \subset X \setminus V$, which contradicts V being nonempty. Hence B must be connected. $\square$

Thm Let $f : X \to Y$ be a continuous function. If X is connected, then so is $f(X)$.

Proof Assume X is connected and suppose $B \subset f(X)$ is clopen. Then the continuity of f implies $f^{-1}(B) \subset X$ is clopen. Since X is connected, either $f^{-1}(B) = \phi$ so that $B = \phi$ or $f^{-1}(B) = X$ so that $B = f(X)$. Thus the only clopen subsets of $f(X)$ are ϕ and $f(X)$, and therefore $f(X)$ is connected. $\square$

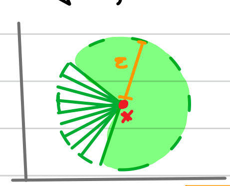
Ex Let us assume, for the moment, that $[0,1) \subset \mathbb{R}$ is connected. For $x, y \in \mathbb{R}^n$ consider $f : [0,1) \to \mathbb{R}^n$ defined by



$$f(t) = (1-t)x + ty = ((1-t)x_1 + ty_1, \dots, (1-t)x_n + ty_n)$$

Then f is continuous since each coordinate function is continuous. The image of f is the line segment from x to y, including x but excluding y, and this is connected by the previous theorem.

Now consider $B_d(x, \varepsilon)$ for $x \in \mathbb{R}^n$, $\varepsilon > 0$ and d the euclidean metric. This ε-ball is the union of line segments from x to y where $\|x - y\| = \varepsilon$. Then intersection of these line segments is nonempty (it contains x). Thus an earlier theorem implies $B_d(x, \varepsilon)$ is connected. $\square$



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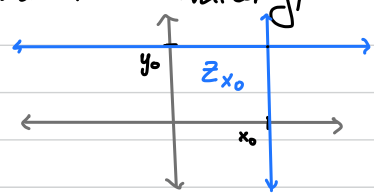
- Since separations are defined via open sets, their existence depends entirely on the topology of the space. Consequently being connected or disconnected also depends entirely on the topology, which means if two spaces are homeomorphic then they are either both connected or both disconnected. We will use this observation in the proof of the next theorem.

Thm If $X_1, \dots, X_n$ are connected topological spaces, then $X_1 \times \cdots \times X_n$ is connected.

Proof We first prove $X \times Y$ is connected for connected topological spaces X and Y. Fix $x_0 \in X$ and $y_0 \in Y$. Then $X \times \{y_0\}$ is homeomorphic to X and therefore connected. Similarly, $\{x_0\} \times Y$ is connected because it is homeomorphic to Y. Note that

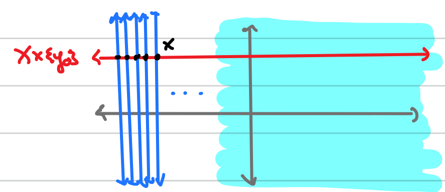


$$(x_0, y_0) \in (X \times \{y_0\}) \cap (\{x_0\} \times Y) \neq \emptyset.$$

Thus an earlier theorem implies

$$Z_{x_0} := (X \times \{y_0\}) \cup (\{x_0\} \times Y)$$

is connected. For any $x \in X$, define Z_x as above (where y_0 remains fixed). Observe that



$$X \times \{y_0\} \subset \bigcap_{x \in X} Z_x \neq \emptyset$$

and

$$\bigcup_{x \in X} Z_x = X \times Y$$

so the same theorem implies $X \times Y$ is connected.

Now we prove the theorem via induction on n. The base case $n=1$ is true by assumption. Suppose we have already shown $X_1 \times \cdots \times X_{n-1}$ is connected. Then letting $X := X_1 \times \cdots \times X_{n-1}$ and $Y := X_n$,

$$X \times Y = X_1 \times \cdots \times X_{n-1} \times X_n$$

is connected by the argument above. Thus induction implies any finite cartesian product of connected spaces is connected. $\square$

- We conclude this section with some examples demonstrating what can happen with infinite products.

EX ① $\mathbb{R}^{\mathbb{N}}$ with the box topology is disconnected. Let $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^{\mathbb{N}}$ be the sets of bounded and unbounded sequences, respectively. Then $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{U} \cup \mathcal{V} = \mathbb{R}^{\mathbb{N}}$. So it remains to show $\mathcal{U}$ and $\mathcal{V}$ are open. For $x \in \mathcal{U}$, $\prod_{n \in \mathbb{N}} (x_n - 1, x_n + 1)$ is a box topology neighborhood of x, and any element in this neighborhood is bounded since x is. Thus this neighborhood is contained in $\mathcal{U}$. Since $x \in \mathcal{U}$ was arbitrary, this shows $\mathcal{U}$ is open. A similar proof shows $\mathcal{V}$ is open (the same type of neighborhood works), and so $\mathcal{U}, \mathcal{V}$ is a separation of $\mathbb{R}^{\mathbb{N}}$.

② $\mathbb{R}^{\mathbb{N}}$ with the product topology is connected. To show this, we must assume $\mathbb{R}$ itself is connected, which will be proven in the next section. For each $N \in \mathbb{N}$, let $C_N \subset \mathbb{R}^{\mathbb{N}}$ be the collection of sequences $(x_n)_{n \in \mathbb{N}}$ with $x_n = 0$ for all $n > N$. Then $(x_n)_{n \in \mathbb{N}} \mapsto (x_1, \dots, x_N)$ defines a homeomorphism from C_N

to $\mathbb{R}^N$ (Exercise check this). The previous theorem implies $\mathbb{R}^N$, and therefore C_N, is connected. Since the constant sequence of all zeros is in C_N for all $N \in \mathbb{N}$, we see that

$$C := \bigcup_{N \in \mathbb{N}} C_N$$

is connected. But C is the set of sequences that are eventually zero, and by Exercise 4.2(b) on Homework 6 $\overline{C} = \mathbb{R}^{\mathbb{N}}$. Thus $\mathbb{R}^{\mathbb{N}}$ is connected by a previous theorem. $\square$

$\boxed{\text{Rem}}$ The above argument can be generalized to show $\prod_{j \in J} X_j$ with the product topology is connected when each X_j is connected and J is arbitrary (even uncountable).

§24 Connected Subspaces of $\mathbb{R}$

- Our examples of connected spaces from last section relied on knowing $\mathbb{R}$ and intervals in $\mathbb{R}$ are connected. We will prove these facts in this section. The proofs only rely on the order structure of $\mathbb{R}$, so we can prove connectedness in greater generality.

Def Let X be a set with more than one element and an order relation $<$. We say X is a <u>linear continuum</u> if

1. every subset $A \subset X$ that is bounded above has a least upper bound (supremum);
2. if $x, y \in X$ satisfy $x < y$, then there exists $z \in X$ with $x < z < y$.

Ex
- $\mathbb{R}$ is a linear continuum and so are $(0,1)$, $[0,1)$, $(0,1]$, and $[0,1]$.
- $[0,1) \cup (1,2]$ is <u>not</u> a linear continuum because $A = [0,1)$ is bounded above by 2 but there is no least upper bound.
- $[0,1] \cup [2,3]$ is <u>not</u> a linear continuum because $1 < 2$ but there is no z satisfying $1 < z < 2$. $\square$

- Recall that we say a subset Y of an ordered set X is convex if whenever $a, b \in Y$ with $a < b$ one has $[a,b] \subset Y$, where

$$[a,b] = \{x \in X \mid a \leq x \leq b\}$$

Thm Let X be a linear continuum equipped with the order topology. Then any convex subset $Y \subset X$ is connected.

Proof Suppose, towards a contradiction, that Y is disconnected, say with separation $U, V \subset Y$. Since U and V are nonempty, we can find $a \in U$ and $b \in V$. Without loss of generality, assume $a < b$. The convexity of Y implies $[a,b] \subset Y$, and we denote:

$$U_0 := U \cap [a,b] \quad \text{and} \quad V_0 := V \cap [a,b].$$

We claim U_0, V_0 is a separation of $[a,b]$ with the subspace topology (which is the same as the order topology since $[a,b]$ is convex). Indeed, U_0 and V_0 are both open in $[a,b]$ and nonempty since $a \in U_0$ while $b \in V_0$. Also

$$U_0 \cap V_0 \subset U \cap V = \emptyset$$

while

$$U_0 \cup V_0 = (U \cap [a,b]) \cup (V \cap [a,b]) = (U \cup V) \cap [a,b] = Y \cap [a,b] = [a,b].$$

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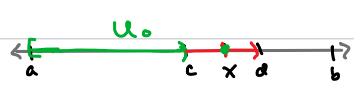
Now, let $c := \sup U_0$. Then $a \leq c \leq b$ since $a \in U_0$ and b is an upper bound for $U_0 \subset [a,b]$. Thus $U_0 \cup V_0 = [a,b]$ implies either $c \in U_0$ or $c \in V_0$. We will arrive at our long sought after contradiction by showing neither of these can hold.

First suppose $c \in U_0$. Since $b \in V_0$, we must therefore have $c < b$. That is c is <u>not</u> the largest element of $[a,b]$. Since U_0 is open and contains c there exists $d \in [a,b]$ with $c < d$ and $[c,d) \subset U_0$. But X being a linear continuum implies there is some $x \in X$ with $c < x < d$. So $x \in [c,d) \subset U_0$, which



contradicts c being an upper bound for U_0. Thus we cannot have $c \in U_0$.
So assume $c \in V_0$. Since $a \in U_0$ we must have $a < c$. So c is not the smallest element of $[a,b]$. Since V_0 is open and contains c, there must be some $d \in [a,b]$ satisfying $d < c$ and $(d, c] \subset V_0$.
Since c is an upper bound for U_0, $[c, b] \subset V_0$.

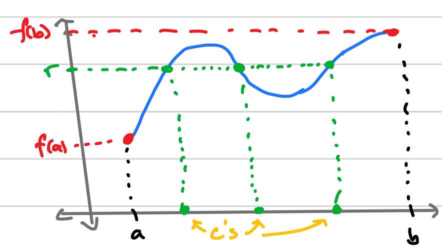
Hence $(d, b] \subset V_0$. But this implies d is an upper bound for U_0, which contradicts c being the least upper bound. $\square$

- We note that X and any interval or ray in X is convex, and hence connected by the previous theorem. In particular,

Cor $\mathbb{R}$ is connected and so are any intervals or rays in $\mathbb{R}$.

Thm (Intermediate Value Theorem)
Let $f: X \to Y$ be a continuous function, where X is a connected topological space and Y is a set with an order relation $<$ and the order topology. If $a, b \in X$ and $r \in Y$ lies between $f(a)$ and $f(b)$ (so $f(a) < r < f(b)$ or $f(b) < r < f(a)$), then there exists $c \in X$ such that $f(c) = r$.

Proof Consider the following disjoint open subsets of $f(X)$:
$$U := (-\infty, r) \cap f(X) \quad \text{and} \quad V := (r, +\infty) \cap f(X).$$
They are non-empty since one contains $f(a)$ and the other $f(b)$. If $U \cup V = f(X)$, then U, V would be a separation for $f(X)$. However, $f(X)$ is connected as the continuous image of a connected set. Thus $U \cup V \subsetneq f(X)$, but
$$Y = (-\infty, r) \cup \{r\} \cup (r, +\infty)$$
implies this is only possible if $r \in f(X)$. That is, there exists $c \in X$ with $f(c) = r$. $\square$



- For $X = [a,b] \subset \mathbb{R}$ and $Y = \mathbb{R}$, the above theorem is the same as the one encounters in calculus/analysis.

Def Let X be a topological space and $x, y \in X$. A path from x to y is a continuous function $f: [a,b] \to X$ with $f(a) = x$ and $f(b) = y$ where $[a,b] \subset \mathbb{R}$ is some closed interval. We say X is path connected if every pair of points $x, y \in X$ admits a path from x to y.

Ex Let d be the euclidean metric on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, $B_d(x, \varepsilon)$ is path connected. Indeed, for $y, z \in B_d(x, \varepsilon)$ define
$$f(t) := (1-t)y + tz \qquad 0 \le t \le 1$$

Note that $f(0)=y$ and $f(1)=z$. Also, the coordinate functions

$$f_j(t) = (1-t)y_j + tz_j \qquad j=1,\dots,n$$

are all continuous, thus f is continuous. We claim that $f(t)\in B_d(x,\varepsilon)$ for all $0\le t\le 1$, so that $f:[0,1]\to B_d(x,\varepsilon)$ is the desired path from x to y. Indeed, we have

$$d(f(t),x) = \|(1-t)y+tz - x\| = \|(1-t)y+tz-(1-t)x-tx\|$$

Homework 7 Exercise 5

$$\left\{\begin{aligned} &= \|(1-t)(y-x)+t(z-x)\| \le \|(1-t)(y-x)\|+\|t(z-x)\| \\ &= |1-t|\cdot\|y-x\| + |t|\cdot\|z-x\| = (1-t)\,d(y,x) + t\,d(z,x)\end{aligned}\right.$$

$$< (1-t)\cdot\varepsilon + t\cdot\varepsilon = \varepsilon.$$

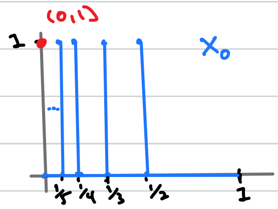
Thus $f(t)\in B_d(x,\varepsilon)$. □

Prop A path connected topological space X is also connected.

Proof Suppose, towards a contradiction, that X is disconnected, say with separation U,V. Let $x\in U$ and $y\in V$. Since X is path connected, there is a path $f:[a,b]\to X$ from x to y. Then $A:=f^{-1}(U)$ and $B:=f^{-1}(V)$ are nonempty open subsets of $[a,b]$ satisfying $A\cap B=\emptyset$ and $A\cup B=[a,b]$. Thus A,B is a separation of $[a,b]$, but this contradicts $[a,b]$ being connected. Therefore X must be connected. □

- The converse of this proposition is not true:

Ex (The Topologist's Comb)



Consider the following subset of $\mathbb{R}^2$:

$$X_0 := ([0,1]\times\{0\}) \cup \bigcup_{n\in\mathbb{N}} \{\tfrac{1}{n}\}\times[0,1]$$

Let $X:=X_0\cup\{(0,1)\}$. Then X is connected but not path connected.

Indeed, we first observe that X_0 is connected since each subset

$X_n := ([0,1]\times\{0\})\cup(\{\tfrac{1}{n}\}\times[0,1])$ is connected (each line segment is connected and their intersection contains $(1/n,0)$) and their intersection contains $[0,1]\times\{0\}$, and therefore

$$X_0 = \bigcup_{n\in\mathbb{N}} X_n$$

is connected. Now, if $U,V\subset X$ was a separation then the connected subspace $X_0\subset X$ satisfies either $X_0\subset U$ or $X_0\subset V$. Assume, without loss of generality, that $X_0\subset U$. Then $(0,1)\in V$, but the sequence $((\tfrac{1}{n},1))_{n\in\mathbb{N}}\subset X_0$ converges to $(0,1)$. Thus $(0,1)\in \overline{U}\cap V\ne\emptyset$, and so U,V cannot be a separation for X. That is, X is connected.

To see that X is not path connected, consider $x:=(1,0)$ and $y:=(0,1)$. Then there is no path from x to y. Suppose, towards a contradiction, that $f:[a,b]\to X$ is a path from x to y. Consider

$$U=(-1,1)\times(\tfrac{1}{2},\tfrac{3}{2}),$$

which is a neighborhood of y. Thus $f^{-1}(U)$ is a neighborhood of b; that is, there exists $\delta>0$ so that $(b-\delta,b]\subset f^{-1}(U)$. Let $t\in(b-\delta,b)$, so that $f(t)\in U\cap X$. If $f(t)\in X_0$ then $f(t)=(\tfrac{1}{n},s)$ for some $n\in\mathbb{N}$ and some $s\in(\tfrac{1}{2},\tfrac{3}{2})$. But this implies

$$f^{-1}((-1,\tfrac{1}{n})\times(\tfrac{1}{2},\tfrac{3}{2})) \quad \text{and} \quad f^{-1}((\tfrac{1}{m},1)\times(\tfrac{1}{2},\tfrac{3}{2}))$$

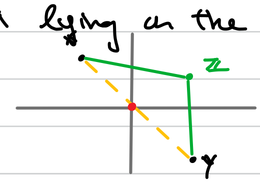
is a separation for $[t,b]$, contradicting the fact that intervals are connected. Thus we must have $f(t)=y$. Therefore $f^{-1}(\{y\}) = (b-\delta, b]$ and so $(b-\delta, b]$ is a clopen set in $[a,b]$. But this contradicts $[a,b]$ being connected. Hence no such path can exist, and therefore X is <u>not</u> path connected. □

- It is often easier to show subsets of $\mathbb{R}^n$ are connected by first showing they are path connected:

Ex ① Let $X := \mathbb{R}^n \setminus \{(0,\dots,0)\}$ (n-dimensional euclidean space without the origin). Then X is path connected and hence connected. Indeed, let $x, y \in \mathbb{R}^n$. If the line connecting x and y does not pass through the origin, then $f:[0,1]\to X$

$$f(t) = ((1-t)x_1+ty_1, \dots, (1-t)x_n+ty_n)$$

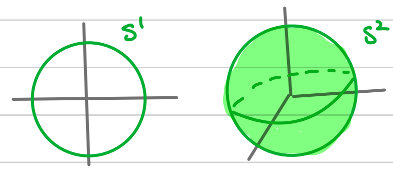
is the desired path. Otherwise let $z \in \mathbb{R}^n$ be a point not lying on the line from x to y. It follows that the lines from x to z and from z to y do not pass through the origin. Hence $g:[0,2]\to X$



$$g(t) = \begin{cases} ((1-t)x_1+tz_1, \dots, (1-t)x_n+tz_n) & \text{if } 0\le t\le 1 \\ ((2-t)z_1+(1-t)y_1, \dots, (2-t)z_n+(1-t)y_n) & \text{if } 1< t\le 2 \end{cases}$$

is the desired path.

② Let $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and let X be as in the previous example. Then $h: X \to S^{n-1}$ defined by



$$h(x) := \frac{1}{\|x\|}x$$

is continuous. Indeed, observe that $\|x\| = d(x, 0)$ where d is the euclidean metric and $0 = (0,\dots,0)$ is the origin. Hence $x \mapsto \|x\|$ is continuous by Exercise 3 on Homework 8, and is nonzero on X. Also the coordinate projection $x \mapsto x_j$ is continuous for each $j=1,\dots,n$. Thus the coordinate functions of h ($x \mapsto \frac{x_j}{\|x\|}$) are quotients of continuous functions and therefore continuous. It follows that h is continuous. Consequently each S^{n-1} is connected as the continuous image of the connected space X. □

§26 Compact Spaces

Def Let X be a top space and $A \subset X$ a subset. An open cover of A is a collection $\mathcal{C}$ of open subsets of X satisfying

$$A \subset \bigcup_{U \in \mathcal{C}} U.$$

When $A = X$, we just call this an open cover. If $S \subset \mathcal{C}$ is a subcollection such that $A \subset \bigcup_{U \in S} U$, then we call S an subcover of $\mathcal{C}$. We say a subcover $S \subset \mathcal{C}$ is finite if S contains finitely many sets: $S = \{U_1, \ldots, U_n\}$.

Ex ① Let $X = \mathbb{R}$ and consider this collection of open sets: $\mathcal{C} := \{(-n, n) \mid n \in \mathbb{N}\}$. Then

$$\bigcup_{U \in \mathcal{C}} U = \bigcup_{n \in \mathbb{N}} (-n, n) = \mathbb{R}$$

So $\mathcal{C}$ is an open cover of $\mathbb{R}$. It is also an open cover for any $A \subset \mathbb{R}$. The subcollection $S := \{(-2n, 2n) \mid n \in \mathbb{N}\} \subset \mathcal{C}$ is a subcover, but it is not finite.

Consider $\mathcal{C}' := \{(n-1, n+1) \mid n \in \mathbb{Z}\}$. Then $\mathcal{C}'$ is an open cover for $\mathbb{R}$ and it has no subcovers besides $\mathcal{C}'$ itself. Indeed, for $n \in \mathbb{Z}$ we have $n \in (n-1, n+1)$ but $n \notin (m-1, m+1)$ for any $m \neq n$. Thus any strict subcollection $S \subsetneq \mathcal{C}$ would not not be a cover for $\mathbb{R}$ since it would miss some $n \in \mathbb{Z}$.

② Let $A := (0, 1] \subset \mathbb{R} =: X$. Then $\mathcal{C}$ and $\mathcal{C}'$ from the previous example are both open covers of A. They both also have finite subcovers: $S := \{(-2, 2)\} \subset \mathcal{C}$ and $S' := \{(0, 2)\} \subset \mathcal{C}'$.

The collection $\mathcal{C}'' := \{(\frac{1}{n}, 2) \mid n \in \mathbb{N}\}$ is also an open cover of A: for any $x \in A$ we have $x > 0$ and so $\exists\, n \in \mathbb{N}$ with $x > \frac{1}{n}$. Thus $x \in (\frac{1}{n}, 2)$. Since $x \in A$ was arbitrary, we have $A \subset \bigcup_{U \in \mathcal{C}''} U$. $S'' := \{(\frac{1}{2n}, 2) \mid n \in \mathbb{N}\} \subset \mathcal{C}''$ is a subcover of $\mathcal{C}''$, but $\mathcal{C}''$ has no finite subcovers. To see this let

$$\{(\tfrac{1}{n_1}, 2), \ldots, (\tfrac{1}{n_d}, 2)\} \subset \mathcal{C}''$$

be a finite subcollection. If $n_0 := \max\{n_1, \ldots, n_d\}$ then

$$\bigcup_{j=1}^{d} (\tfrac{1}{n_j}, 2) = (\tfrac{1}{n_0}, 2) \not\supset (0, 1]. \qquad \square$$

Def Let X be a topological space. We say a subset $A \subset X$ is compact if every open cover of A has a finite subcover. If $A = X$, we call X a compact space.

Ex ① $\mathbb{R}$ is not compact: both collections $\mathcal{C}$ and $\mathcal{C}'$ in the previous example failed to have finite subcovers. In fact $\mathcal{C}'$ had no subcovers at all.

② $(0, 1] \subset \mathbb{R}$ is not compact because $\mathcal{C}''$ had no finite subcover.

③ Let X be any topological space. If $A \subset X$ contains finitely many points then A is compact. (Exercise check this.)

4 $A := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is compact. Let $\mathcal{C}$ be an open cover of A. Then $\exists\, \mathcal{U}_0 \in \mathcal{C}$ with $0 \in \mathcal{U}_0$. Since $\mathcal{U}$ is open $\exists\, \varepsilon > 0$ so that $(-\varepsilon, \varepsilon) \subset \mathcal{U}_0$. Consequently, if $N \in \mathbb{N}$ is such that $\frac{1}{N} \leq \varepsilon$ then $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subset \mathcal{U}_0$ for all $n \geq N$. That is $\mathcal{U}_0$ contains all but finitely many points of A. For $n \in \{1, 2, \ldots, N\}$, let $\mathcal{U}_n \in \mathcal{C}$ be such that $\frac{1}{n} \in \mathcal{U}_n$. Then $\{\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_N\}$ is a finite subcover. Since $\mathcal{C}$ was an arbitrary open cover of A, we see that A is compact. $\square$

- In Example 2 above we showed $(0,1]$ was not a compact subset of $\mathbb{R}$. It is natural to wonder if this is still the case when we view $(0,1]$ as the space X itself. That is, if we require our open covers to consist of open subsets of $(0,1]$ (rather than $\mathbb{R}$). In this case we still have that $(0,1]$ is not compact: $\{(\frac{1}{n}, 1] \mid n \in \mathbb{N}\}$ is an open cover of $(0,1]$ (in $(0,1]$) and it has no finite subcover by the same argument as in Example 2. The next lemma shows this is true in general: the compactness of $A \subset X$ is independent of whether we view A as a subset of X or a topological space in its own right.

Lemma Let X be a topological space and $Y \subset X$ a subspace. Then $A \subset Y$ is compact in Y if and only if A is compact in X. In particular, Y is a compact space if and only if Y is a compact subset of X.

Proof ($\Longrightarrow$): Assume A is a compact subset of Y. Let $\mathcal{C}$ be an open cover for A consisting of open subsets of X. Then $\mathcal{C}' = \{\mathcal{U} \cap Y \mid \mathcal{U} \in \mathcal{C}\}$ consists of open subsets of Y and

$$\bigcup_{V \in \mathcal{C}'} V = \bigcup_{\mathcal{U} \in \mathcal{C}} \mathcal{U} \cap Y = \Big(\bigcup_{\mathcal{U} \in \mathcal{C}} \mathcal{U}\Big) \cap Y \supset A \cap Y = A$$

so $\mathcal{C}'$ is an open cover of A in Y. By compactness there exists a finite subcover $\{\mathcal{U}_1 \cap Y, \ldots, \mathcal{U}_n \cap Y\} \subset \mathcal{C}'$. Then

$$\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n \supset (\mathcal{U}_1 \cap Y) \cup \cdots \cup (\mathcal{U}_n \cap Y) \supset A,$$

so $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ is a finite subcover of $\mathcal{C}$. Thus A is compact in X.

($\Longleftarrow$): Assume A is a compact set in X. Let $\mathcal{C}$ be an open cover of A consisting of sets that are open in Y. For each $\mathcal{U} \in \mathcal{C}$, there exists $V \subset X$ open in X with $\mathcal{U} = V \cap Y$. Let $\mathcal{C}'$ be the collection of all such V. Then $\mathcal{C}'$ is an open cover for A in X:

$$\bigcup_{V \in \mathcal{C}'} V \supset \bigcup_{V \in \mathcal{C}'} V \cap Y = \bigcup_{\mathcal{U} \in \mathcal{C}} \mathcal{U} \supset A.$$

By compactness of A in X, there exists a finite subcover $\{V_1, \ldots, V_n\} \subset \mathcal{C}'$. But then $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcover of $\mathcal{C}$:

$$(V_1 \cap Y) \cup \cdots \cup (V_n \cap Y) = (V_1 \cup \cdots \cup V_n) \cap Y \supset A \cap Y = A.$$

Hence A is compact in Y.

The last statement follows by taking $A = Y$. $\square$

- The previous lemma tells us it is not necessary to say subsets are compact *in* X/Y. Rather, the compactness of a set depends only on the set itself (and the subspace topology it inherits from the ambient space).

Thm If X is a compact space and $A \subset X$ is closed, then A is compact.

Proof Let $\mathcal{C}$ be an open cover of A in X. Since A is closed, $X \setminus A$ is open and $\mathcal{C}' := \{X \setminus A\} \cup \mathcal{C}$ is an open cover of X:

$$\bigcup_{U \in \mathcal{C}'} U = (X \setminus A) \cup \bigcup_{U \in \mathcal{C}} \supset (X \setminus A) \cup A = X$$

Since X is compact, there is a finite subcover $\{U_1, \ldots, U_n\} \subset \mathcal{C}'$:

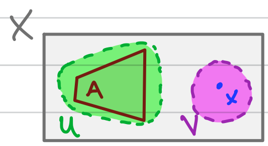
$$U_1 \cup U_2 \cup \cdots \cup U_n \supset X$$

If $U_j = X \setminus A$ for some $j = 1, \ldots, n$, say — without loss of generality — $j = n$, then

$$U_1 \cup U_2 \cup \cdots \cup U_{n-1} \supset X \setminus U_n = X \setminus (X \setminus A) = A$$

Hence $\{U_1, \ldots, U_{n-1}\} \subset \mathcal{C}$ is a finite subcover of $\mathcal{C}$. If $U_j \neq X \setminus A$ for all $j = 1, \ldots, n$, then $\{U_1, \ldots, U_n\} \subset \mathcal{C}$ is a finite subcover of $\mathcal{C}$. Thus in any case $\mathcal{C}$ has a finite subcover of A and so A is compact. $\square$

Lemma Let X be a Hausdorff space. If $A \subset X$ is compact and $x \in X \setminus A$, then there exist disjoint open sets $U, V \subset X$ with $A \subset U$ and $x \in V$.



Proof Since X is Hausdorff, for each $a \in A$ there exists disjoint open neighborhoods U_a and V_a for a and x, respectively. Observe that $\mathcal{C} := \{U_a \mid a \in A\}$ is an open cover of A:

$$\bigcup_{U \in \mathcal{C}} U = \bigcup_{a \in A} U_a \supset \bigcup_{a \in A} \{a\} = A.$$

Thus the compactness of A implies there is a finite subcover $\{U_{a_1}, \ldots, U_{a_n}\} \subset \mathcal{C}$. Define

$$U := U_{a_1} \cup \cdots \cup U_{a_n} \quad \text{and} \quad V := V_{a_1} \cap \cdots \cap V_{a_n}.$$

Then $A \subset U$ since $\{U_{a_1}, \ldots, U_{a_n}\}$ is a subcover. V is an open set and containing x since $x \in V_{a_j}$ for $j = 1, \ldots, n$. Finally

$$\begin{aligned} U \cap V &= (U_{a_1} \cap V) \cup \cdots \cup (U_{a_n} \cap V) \\ &\subset (U_{a_1} \cap V_{a_1}) \cup \cdots \cup (U_{a_n} \cap V_{a_n}) \\ &= \emptyset \cup \cdots \cup \emptyset = \emptyset. \end{aligned}$$

Thus U and V are disjoint. $\square$

- The proof of the next theorem should remind you of our proof that singleton sets in Hausdorff spaces are closed.

Thm Let X be a Hausdorff space. If $A \subset X$ is compact, then A is closed.

Proof We will show $X \setminus A$ is open, and will do so by finding for each $x \in X \setminus A$ a neighborhood V satisfying $V \subset X \setminus A$. Fix $x \in X \setminus A$. By the previous lemma there exist disjoint open subsets $U, V \subset X$ satisfying $A \subset U$ and $x \in V$. Consequently,

$$V \subset X \setminus U \subset X \setminus A$$

and so V is the desired neighborhood. $\square$

Ex ① Using the contrapositive of the previous theorem, we can conclude that the intervals (a,b), $(a,b]$, $[a,b) \subset \mathbb{R}$ are not compact since they are not closed. However, we will later see that $[a,b]$ is compact.

② The hypothesis that X is Hausdorff is needed in the previous theorem. Equip $\mathbb{R}$ with the finite complement topology, which is not Hausdorff by Exercise 3 on Homework 5. Then every subset $A \subset \mathbb{R}$ is compact. However, $A \subset \mathbb{R}$ is closed iff $\mathbb{R} \setminus A$ is open iff $\mathbb{R} \setminus (\mathbb{R} \setminus A) = A$ is finite or all of $\mathbb{R}$. Thus any infinite subset $A \subsetneq \mathbb{R}$ is compact but not closed. $\square$

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Thm Let $f: X \to Y$ be a continuous function. If $A \subset X$ is compact, then $f(A) \subset Y$ is compact.

Proof Let $\mathcal{C}$ be an open cover of $f(A)$. Consider

$$\mathcal{C}' := \{ f^{-1}(V) \mid V \in \mathcal{C} \}.$$

Since f is continuous, $\mathcal{C}'$ consists of open subsets of X. Moreover

$$\bigcup_{U \in \mathcal{C}'} U = \bigcup_{V \in \mathcal{C}} f^{-1}(V) = f^{-1}\Big(\bigcup_{V \in \mathcal{C}} V\Big) \supset f^{-1}(f(A)) \supset A,$$

where the last containment follows from Exercise 1.(a) on Homework 1. Thus $\mathcal{C}'$ is an open cover of A. Using compactness, let $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\} \subset \mathcal{C}'$ be a finite subcover. We claim $\{V_1, \ldots, V_n\} \subset \mathcal{C}$ is a finite subcover. Indeed, if $y \in f(A)$ then there exists $x \in A$ with $f(x) = y$. Then $x \in f^{-1}(V_j)$ for some $j = 1, \ldots, n$, and hence $y = f(x) \in V_j$. Thus

$$f(A) \subset V_1 \cup \cdots \cup V_n$$

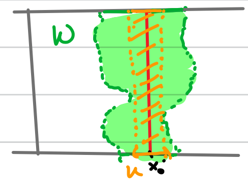
and so $\{V_1, \ldots, V_n\}$ is a finite subcover of $\mathcal{C}$. Hence $f(A)$ is compact. $\square$

- The above theorem tells us continuous functions send compact sets to compact sets, just like with connected sets. This is the opposite of open sets and closed sets (which are pulled back to open/closed sets by continuous functions). This difference is vital to the proof of the next theorem.

Thm Let $f: X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof Denote $g := f^{-1}$. It suffices to show g is continuous. Let $A \subset X$ be closed. By the first theorem in this section, A is compact. By the previous theorem $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed. Thus $g^{-1}(A) = f(A)$ is closed for all closed subsets $A \subset X$. Therefore g is continuous. $\square$

Lemma (The tube lemma)



Let X and Y be topological spaces with Y compact, and give $X \times Y$ the product topology. If $W \subset X \times Y$ is open and $\{x_0\} \times Y \subset W$ for some $x_0 \in X$, then $U \times Y \subset W$ for some neighborhood $U \subset X$ of x_0.

Proof We first observe that $\{x_0\} \times Y$ is homeomorphic to Y and hence compact. Now, recall that sets of the form $U \times V$, where $U \subset X$ and $V \subset Y$ are open, form a basis for the product topology. Thus for all $y \in Y$ there exists open sets $U_y \subset X$ and $V_y \subset Y$ satisfying

$$(x_0, y) \subset U_y \times V_y \subset W$$

Note that

$$\bigcup_{y \in Y} U_y \times V_y \supset \bigcup_{y \in Y} \{(x_0, y)\} = \{x_0\} \times Y.$$

Thus $\{U_y \times V_y \mid y \in Y\}$ is an open cover for $\{x_0\} \times Y$. Compactness of Y yields a finite subcover $\{U_{y_1} \times V_{y_1}, \dots, U_{y_n} \times V_{y_n}\}$. Set

$$U := \bigcap_{j=1}^{n} U_{y_j}$$

Then U is open since it is a finite intersection of open sets. Also $x_0 \in U$ since $x_0 \in U_{y_j}$ for each $j = 1, \dots, n$. So U is a neighborhood of x_0 and

$$U \times Y \subset \bigcup_{j=1}^{n} U \times V_j \subset \bigcup_{j=1}^{n} U_j \times V_j \subset W. \qquad \square$$

Thm If $X_1, \dots, X_n$ are compact, then $X_1 \times \cdots \times X_n$ is compact.

Proof We first show $X \times Y$ is compact when X, Y are compact. Let $\mathcal{C}$ be an open cover for $X \times Y$. Fix $x \in X$. Then $\{x\} \times Y$ is homeomorphic to Y and therefore compact. Since $\mathcal{C}$ is also an open cover for $\{x\} \times Y$, compactness yields a finite subcover $\{W_1, \dots, W_n\} \subset \mathcal{C}$. Thus

$$\{x\} \times Y \subset W_1 \cup \cdots \cup W_n$$

and so the tube lemma implies there is a neighborhood $U_x \subset X$ of x so that

$$U_x \times Y \subset W_1 \cup \cdots \cup W_n.$$

Denote $\mathcal{C}_x := \{W_1, \dots, W_n\}$. Now, $\{U_x \mid x \in X\}$ is an open cover for X, so compactness gives a finite subcover $\{U_{x_1}, \dots, U_{x_d}\}$. Note that $\bigcup_{j=1}^{d} \mathcal{C}_{x_j} \subset \mathcal{C}$ is finite and

$$\bigcup_{j=1}^{d} \bigcup_{W \in \mathcal{C}_{x_j}} W \supset \bigcup_{j=1}^{d} U_{x_j} \times Y = \left(\bigcup_{j=1}^{d} U_{x_j}\right) \times Y \supset X \times Y$$

Thus $\bigcup_{j=1}^{d} \mathcal{C}_{x_j}$ is a finite subcover and $X \times Y$ is compact.

Now, we prove the theorem by induction on n. The base case $n=1$ is true by assumption. Suppose we have already shown $X_1 \times \cdots \times X_{n-1}$ is compact. Then letting $X := X_1 \times \cdots \times X_{n-1}$ and $Y := X_n$,

$$X \times Y = X_1 \times \cdots \times X_{n-1} \times X_n$$

is compact by the argument above. Thus induction implies any finite cartesian product of compact spaces is compact. $\square$

- What about infinite cartesian products of compact sets? It turns out, by Tychonoff's theorem, that these spaces are also compact.

- We conclude the section with two ways to characterize compactness

Def We say a collection $\mathcal{C}$ of subsets of X has the finite intersection property if for every finite subcollection $\{C_1, \ldots, C_n\} \subset \mathcal{C}$ one has
$$C_1 \cap \cdots \cap C_n \neq \emptyset$$

Thm A topological space X is compact if and only if every collection $\mathcal{C}$ of closed subsets of X with the finite intersection property satisfies
$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset$$

Proof $(\Longrightarrow)$: Assume X is compact and let $\mathcal{C}$ be a collection of closed subsets of X with the finite intersection property. Suppose, towards a contradiction, that $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Consequently
$$X = X \setminus \Big(\bigcap_{C \in \mathcal{C}} C \Big) = \bigcup_{C \in \mathcal{C}} (X \setminus C)$$
Since C is closed, $X \setminus C$ is open, and so the above shows $\{X \setminus C \mid C \in \mathcal{C}\}$ is an open cover of X. Compactness of X then yields a finite subcover: $\{X \setminus C_1, \ldots, X \setminus C_n\}$. So
$$X = (X \setminus C_1) \cup \cdots \cup (X \setminus C_n) = X \setminus (C_1 \cap \cdots \cap C_n),$$
which implies $C_1 \cap \cdots \cap C_n = \emptyset$, contradicting the finite intersection property of $\mathcal{C}$. Thus we must have $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

$(\Longleftarrow)$: We proceed by a contrapositive. Assume X is not compact. Then there exists an open cover $\mathcal{D}$ of X with no finite subcover. That is,
$$\bigcup_{U \in \mathcal{D}} U = X \qquad (1)$$
but for any finite subcollection $\{U_1, \ldots, U_n\} \subset \mathcal{D}$ one has
$$U_1 \cup \cdots \cup U_n \subsetneq X \qquad (2)$$
Consequently, if we define $\mathcal{C} := \{X \setminus U \mid U \in \mathcal{D}\}$, then $\mathcal{C}$ is a collection of closed subsets of X. Then (2) implies for any finite subcollection $\{X \setminus U_1, \ldots, X \setminus U_n\}$
$$(X \setminus U_1) \cap \cdots \cap (X \setminus U_n) = X \setminus (U_1 \cup \cdots \cup U_n) \neq \emptyset.$$
That is, $\mathcal{C}$ has the finite intersection property. However, (1) implies
$$\bigcap_{C \in \mathcal{C}} C = \bigcap_{U \in \mathcal{D}} X \setminus U = X \setminus \Big(\bigcup_{U \in \mathcal{D}} U \Big) = \emptyset$$
$\square$

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- Let $\{C_n \mid n \in \mathbb{N}\}$ be a sequence of nonempty subsets of X satisfying
$$C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots.$$
Then $C_{n_1} \cap \cdots \cap C_{n_m} = C_k$ where $k = \max\{n_1, \ldots, n_m\}$. Thus $\{C_n \mid n \in \mathbb{N}\}$ has the finite intersection property. If X is compact and each C_n is closed, then the theorem implies $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

However, this is not true for non-compact spaces:

Ex Consider $X := (0,1]$, which we have seen is not compact, and the closed subsets $C_n := (0, \frac{1}{n}]$ for $n \geq 1$. Then $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$, and so the collection $\mathcal{C} := \{C_n \mid n \in \mathbb{N}\}$ has the finite intersection property. However,

$$\bigcap_{n \in \mathbb{N}} C_n = \bigcap_{n \in \mathbb{N}} (0, \tfrac{1}{n}] = \phi$$

since for any $x \in (0,1]$ there exists $n \in \mathbb{N}$ with $\frac{1}{n} < x$. □

• The previous theorem will enable us to characterize compactness in terms of convergent "subnets":

Def Let $(x_i)_{i \in I}$ be a net over a directed set I. A net $(y_j)_{j \in J}$ over a directed set J is a <u>subnet</u> of $(x_i)_{i \in I}$ if there exists a map $\sigma : J \to I$ such that

1. $y_j = x_{\sigma(j)}$ for all $j \in J$
2. $j_1 \leq j_2$ implies $\sigma(j_1) \leq \sigma(j_2)$ (<u>monotone</u>)
3. for any $i \in I$ there exists $j \in J$ such that $\sigma(j) \geq i$. (<u>final</u>)

Ex Let $(x_n)_{n \in \mathbb{N}}$ be a sequence. Then any subsequence $(x_{n_k})_{k \in \mathbb{N}}$ is a subnet, where $\sigma : \mathbb{N} \to \mathbb{N}$ is defined by $\sigma(k) = n_k$. However, not all subnets of $(x_n)_{n \in \mathbb{N}}$ are subsequences. For example, $(x_1, x_1, x_1, x_4, x_5, \ldots)$ is a valid subnet but not a valid subsequence □

Thm Let X be a topological space. Then X is compact if and only if every net $(x_i)_{i \in I} \subset X$ has a convergent subnet

Proof ($\Rightarrow$) Assume X is compact and let $(x_i)_{i \in I}$ be a net. For each $i \in I$, define $C_i := \overline{\{x_j : j \geq i\}}$. Then for $F \subset I$ finite, we can find $k \in I$ satisfying $k \geq i$ for all $i \in F$. Consequently

$$x_k \in \bigcap_{i \in F} C_i \neq \phi$$

Hence $\mathcal{C} = \{C_i \mid i \in I\}$ is a collection of closed sets with the finite intersection property. Since X is compact, it follows that

$$\bigcap_{i \in I} C_i \neq \phi$$

Let y be an element of this set. Then for every $i \in I$, $y \in C_i = \overline{\{x_j \mid j \geq i\}}$ and so for every neighborhood U of y, $U \cap \{x_j \mid j \geq i\} \neq \phi$. That is, for every $i \in I$ and every neighborhood U of y, there exists $j \geq i$ with $x_j \in U$. Set $y(U,i,j) := x_j$. Then $(y(U,i,j))_{(U,i,j)}$ is a net (where $(U,i,j) \leq (U',i',j')$ iff $U \supset U'$, $i \leq i'$, and $j \leq j'$). In fact, it is a subnet of $(x_i)_{i \in I}$ with $\sigma(U,i,j) = j$:

1. $y(U,i,j) = x_j = x_{\sigma(U,i,j)}$
2. $(U,i,j) \leq (U',i',j')$ implies $\sigma(U,i,j) = j \leq j' = \sigma(U',i',j')$
3. for $i \in I$, $\sigma(U,i,j) = j \geq i$.

Moreover, this subnet converges to y: for a neighborhood $\mathcal{U}$ of y, if $(\mathcal{U}', i', j') \geq (\mathcal{U}, i, j)$ then

$$y_{(\mathcal{U}', i', j')} \in \mathcal{U}' \subset \mathcal{U}.$$

Thus $(x_i)_{i \in I}$ has a convergent subnet.

($\Longleftarrow$) Assume every net has a convergent subnet and let $\mathcal{C}$ be an open cover of X. Suppose, towards a contradiction, that $\mathcal{C}$ has no finite subcover. Let $\mathcal{F} := \{ S \subset \mathcal{C} \mid S \text{ is finite} \}$, which we make into a directed set with $S \leq S'$ iff $S \subset S'$. Since $\mathcal{C}$ has no finite subcovers, $X \not\subset \bigcup_{U \in S} U$ for any $S \in \mathcal{F}$. For each $S \in \mathcal{F}$, choose $x_S \in X \setminus \bigcup_{U \in S} U$. Observe that if $S' \geq S$, then $S' \supset S$ implies $\bigcup_{U \in S'} U \supset \bigcup_{U \in S} U$ and so

$$x_{S'} \in X \setminus \Big(\bigcup_{U \in S'} U \Big) \subset X \setminus \Big(\bigcup_{U \in S} S \Big)$$

Now, $(x_S)_{S \in \mathcal{F}}$ is a net and consequently has a convergent subnet $(x_{\sigma(j)})_{j \in J}$, say with limit $x \in X$. Since $\mathcal{C}$ is an open cover for X, $x \in \mathcal{U}$ for some $\mathcal{U} \in \mathcal{C}$. Then there exists $j_0 \in J$ so that $x_{\sigma(j)} \in \mathcal{U}$ for all $j \geq j_0$. By finality of the subnet, there exists $j_1 \in J$ so that $\sigma(j_1) \geq \{\mathcal{U}\}$. Since J is a directed set, there exists $k \in J$ satisfying $k \geq j_0$ and $k \geq j_1$. Thus $x_{\sigma(k)} \in \mathcal{U}$, but $\sigma(k) \geq \sigma(j_1) \geq \{\mathcal{U}\}$ implies $x_k \in X \setminus \mathcal{U}$, a contradiction. So $\mathcal{C}$ must have a finite subcover, and hence X is compact. $\square$

§ 27 Compact Subspaces of the Real Line

- In this section, we will show closed intervals in $\mathbb{R}$ are compact. The results of the previous section then enable us to show closed bounded subsets of $\mathbb{R}^n$ are compact.

Thm Let X be an ordered set with the least upper bound property. In the order topology, a closed interval $[a,b] \subset X$ is compact.

Proof Let $\mathcal{C}$ be an open cover of $[a,b]$. Consider

$$A := \{x \in [a,b] \mid [a,x] \text{ is covered by finitely many sets in } \mathcal{C}\}.$$

We will show $b \in A$ and thus $[a,b]$ is compact. We first require a claim:

Claim For any $x \in [a,b)$, there exists $y \in (x,b]$ so that $[x,y]$ can be covered by two or less sets in $\mathcal{C}$.

Indeed, let $U_1 \in \mathcal{C}$ be such that $x \in U_1$. Since U_1 is open and $x < b$, there exists $y \in (x,b]$ such that $[x,y) \subset U_1$. Let $U_2 \in \mathcal{C}$ be such that $y \in U_2$. Then $[x,y] \subset U_1 \cup U_2$ □

Applying the claim to $x = a$ shows that there exists $y \in A$ with $y > a$. Now, A is bounded by b (since it is a subset of $[a,b]$) and so the least upper bound property tells us $c := \sup A$ exists and satisfies $a < c \leq b$.

Now, let $U \in \mathcal{C}$ contain c. Then there exists $d < c$ satisfying $(d,c] \subset U$. There must be some $x \in A \cap (d,c]$, because otherwise d would be an upper bound for A that is strictly less than c. So $[a,x]$ can be covered by some $U_1, \ldots, U_n \in \mathcal{C}$, and consequently $[a,c] = [a,x] \cup [x,c]$ can be covered by $U_1, \ldots, U_n, U \in \mathcal{C}$. Thus $c \in A$.

Finally, suppose, towards a contradiction, that $c < b$. Applying the claim to $x = c$ yields $y \in (c,b]$ so that $[c,y]$ can be covered by one or two elements of $\mathcal{C}$. Since $c \in A$, we then see that $[a,y] = [a,c] \cup [c,y]$ can be covered by finitely elements of $\mathcal{C}$. Hence $y \in A$, but $y > c$ contradicts c being an upper bound for A. Thus we must have $c = b$. □

- Since $\mathbb{R}$ with its usual ordering has the least upper bound property, we immediately obtain:

Cor Closed intervals in $\mathbb{R}$ are compact.

- Recall that in a metric space (X,d) we say $A \subset X$ is bounded if there exists $M > 0$ so that $d(a,b) \leq M$ for all $a,b \in A$.

Thm A subset $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded with respect to either the euclidean metric d or the square metric ρ.

Proof Recall that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\rho(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{y}) \leq \sqrt{n}\,\rho(\mathbf{x},\mathbf{y})$$

So a set is bounded with respect to d if and only if it bounded with respect to ρ.

this means we can (and will) work only with ρ.

$(\Rightarrow)$: Suppose $A \subset \mathbb{R}^n$ is compact. Since $\mathbb{R}^n$ is Hausdorff, A is closed. To see that it is bounded consider $\mathcal{C} := \{ B_\rho(\mathbf{0}, n) \mid n \in \mathbb{N} \}$ where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$. Note that

$$B_\rho(\mathbf{0}, n) = (-n, n) \times \cdots \times (-n, n)$$

Thus $\mathcal{C}$ is an open cover for A (in fact an open cover for $\mathbb{R}^n$). Compactness yields a finite subcover $\{ B_\rho(\mathbf{0}, n_1), \dots, B_\rho(\mathbf{0}, n_\ell) \}$. Hence

$$A \subset B_\rho(\mathbf{0}, n_1) \cup \cdots \cup B_\rho(\mathbf{0}, n_\ell) = B_\rho(\mathbf{0}, N)$$

where $N = \max\{n_1, \dots, n_\ell\}$. Thus A is bounded with respect to ρ.

$(\Leftarrow)$: Suppose $A \subset \mathbb{R}^n$ is closed and bounded with respect to ρ. Then there exists $M > 0$ so that $\rho(\mathbf{x}, \mathbf{y}) \leq M$ for all $\mathbf{x}, \mathbf{y} \in A$. Fix $\mathbf{x} = (x_1, \dots, x_n) \in A$. Then $\rho(\mathbf{x}, \mathbf{y}) \leq M$ implies $|x_j - y_j| \leq M$ for each $j = 1, \dots, n$. Thus

$$A \subseteq [x_1 - M, x_1 + M] \times \cdots \times [x_n - M, x_n + M]$$

Each $[x_j - M, x_j + M]$ is compact by the previous corollary. By a theorem from the previous section, their (finite) cartesian product is compact. Hence A is a closed subset of a compact set, and is therefore compact itself by another theorem from the previous section. $\square$

Rem Remember that boundedness of a set depends on the metric, whereas compactness depends only on the topology. Thus the above theorem need not hold for arbitrary metric spaces. For example, if $\bar{d}(x, y) = \min\{|x - y|, 1\}$ is the standard bounded metric on $\mathbb{R}$, then $\mathbb{R}$ is closed and bounded with respect to $\bar{d}$ since $\bar{d}(x, y) \leq 1$ for all $x, y \in \mathbb{R}$. However, we have already seen $\mathbb{R}$ is not compact in the standard topology, which is precisely the topology induced by $\bar{d}$. $\square$

Thm (Extreme value theorem)
Let X be a topological space, let Y be an ordered set with the order topology, and let $f: X \to Y$ be a continuous function. If $A \subset X$ is compact, then there exists $a, b \in A$ so that for all $x \in A$

$$f(a) \leq f(x) \leq f(b)$$

Proof Since A is compact, so is $f(A) \subseteq Y$. We will show $f(A)$ has a smallest element c and a largest element d: $c \leq y \leq d$ for all $y \in f(A)$. But then $c, d \in f(A)$ implies there are $a, b \in A$ satisfying $f(a) = c$ and $f(b) = d$.

Suppose, towards a contradiction, that $f(A)$ has no largest element. Then for all $y \in f(A)$, there exists $z \in f(A)$ with $z > y$. That is, $y \in (-\infty, z)$. Consequently, $\{ (-\infty, z) \mid z \in f(A) \}$ is an open cover for $f(A)$. Then compactness gives us a finite subcover: $\{ (-\infty, z_1), \dots, (-\infty, z_n) \}$. Let $z_0 := \max\{z_1, \dots, z_n\}$. Then $z_0 \notin (-\infty, z_j)$ for any $j = 1, \dots, n$ but $z_0 \in f(A)$ since $z_1, \dots, z_n \in f(A)$. Thus we have a contradiction, and so $f(A)$ must have a largest element.

A similar argument shows $f(A)$ has a smallest element as well. $\square$

Compactness in Metric Spaces

- We explore some consequences of compactness for metric spaces.

Def Let (X,d) be a metric space. For $x \in X$ and a nonempty $A \subset X$, the distance from x to A is the quantity $d(x,A) := \inf_{a \in A} d(x,a)$.

Exercise Show $d(x,A) = 0$ iff $x \in \bar{A}$.

Lemma Let (X,d) be a metric space with subset $A \subset X$. Then $x \mapsto d(x,A)$ is continuous.

Proof For $x, y \in X$ observe that for each $a \in A$

$$d(x,A) \le d(x,a) \le d(x,y) + d(y,a)$$

Thus taking an infimum over $a \in A$ on the right we obtain

$$d(x,A) \le d(x,y) + d(y,A)$$

or $d(x,A) - d(y,A) \le d(x,y)$. Reversing the roles of x and y yields

$$d(y,A) - d(x,A) \le d(y,x) = d(x,y).$$

Thus $|d(x,A) - d(y,A)| \le d(x,y)$, and so the function satisfies the ε-δ definition of continuity with $\delta = \varepsilon$. $\square$

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- Recall that the diameter of $A \subset X$ is $\operatorname{diam}(A) = \sup_{a,b \in A} d(a,b)$.

Lemma (The Lebesgue number lemma)

Let (X,d) be a compact metric space. If $\mathcal{C}$ is an open cover for X, then there exists $\delta > 0$ so that whenever $A \subset X$ satisfies $\operatorname{diam}(A) < \delta$ one has $A \subset U$ for some $U \in \mathcal{C}$.

Proof Let $\mathcal{C}$ be an open cover for X. Compactness gives a finite subcover $\{U_1, \dots, U_n\} \subset \mathcal{C}$. Define $f: X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{j=1}^{n} d(x, X \setminus U_j)$$

Each $d(\cdot, X \setminus U_j)$ is continuous by the lemma, so f is continuous as their sum (times a constant). Moreover, $f(x) > 0$ for all $x \in X$. Indeed, for $x \in X$ we have $x \in U_j$ for some $j = 1, \dots, n$. Since U_j is open, there exists $\varepsilon > 0$ so that $B_d(x, \varepsilon) \subset U_j$. Thus for any $y \in X \setminus U_j$ we must have $d(x,y) \ge \varepsilon$. Consequently

$$f(x) \ge \frac{1}{n} d(x, X \setminus U_j) \ge \frac{1}{n} \varepsilon > 0.$$

Now, by the extreme value theorem, $f(X)$ has a minimal element δ. Since $f(X) \subset (0, +\infty)$, we must have $\delta > 0$. We claim this is the desired number.

Let $A \subset X$ with $\operatorname{diam}(A) < \delta$. Fix $x_0 \in A$, then $A \subset B_d(x_0, \delta)$. Let $1 \le k \le n$ be such that $d(x_0, X \setminus U_k) = \max\{d(x_0, X \setminus U_1), \dots, d(x_0, X \setminus U_n)\}$. Then

$$\delta \le f(x_0) = \frac{1}{n} \sum_{j=1}^{n} d(x_0, X \setminus U_j) \le d(x_0, X \setminus U_k)$$

Thus there are no $y \in X \setminus U_k$ satisfying $d(x_0, y) < \delta$. Thus

$$A \subset B_d(x_0, \delta) \subset X \setminus (X \setminus U_k) = U_k \in \mathcal{C}$$

$\square$

Def The $\delta > 0$ in the above lemma is called a Lebesgue number of the open cover $\mathcal{C}$.

Def Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ so that for any $x_1, x_2 \in X$

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

- The key difference with continuity, is that for uniform continuity δ must not depend on the inputs x_1 or x_2.

Ex ① The function $x \mapsto d(x, A)$ was uniformly continuous since we can always pick $\delta = \varepsilon$ regardless of the inputs.

② The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not uniformly continuous. Let $\varepsilon = 1$. Then for any $\delta > 0$ if $y = x + \frac{\delta}{2}$ we have $|x - y| < \delta$ but

$$|f(x) - f(y)| = |x^2 - (x^2 + \delta x + \tfrac{\delta^2}{4})| = |-\delta x - \tfrac{\delta^2}{4}| \geq \delta |x| - \tfrac{\delta^2}{4}$$

Thus for $x = (1 + \delta^2/4)/2\delta$ we have $|f(x) - f(y)| \geq 1$. □

Thm (Uniform continuity theorem)
Let (X, d_X) be a compact metric space and (Y, d_Y) a metric space. If $f: X \to Y$ is continuous, then it is uniformly continuous.

Proof Let $\varepsilon > 0$. Consider

$$\mathcal{C} := \{ f^{-1}(B_{d_Y}(y, \tfrac{\varepsilon}{2})) \mid y \in Y \}.$$

Then $\mathcal{C}$ is an open cover because f is continuous and

$$\bigcup_{U \in \mathcal{C}} U = \bigcup_{y \in Y} f^{-1}(B_{d_Y}(y, \tfrac{\varepsilon}{2})) = f^{-1}\Big(\bigcup_{y \in Y} B_{d_Y}(y, \tfrac{\varepsilon}{2})\Big) = f^{-1}(Y) = X.$$

Let $\delta > 0$ be the Lebesgue number of $\mathcal{C}$. Thus if $x_1, x_2 \in X$ satisfy $d_X(x_1, x_2) < \delta$, then $\operatorname{diam}\{x_1, x_2\} < \delta$. Consequently, $\{x_1, x_2\} \in f^{-1}(B_{d_Y}(y, \tfrac{\varepsilon}{2}))$ for some $y \in Y$. That is, $f(x_1), f(x_2) \in B_{d_Y}(y, \tfrac{\varepsilon}{2})$. Hence

$$d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), y) + d_Y(y, f(x_2)) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Therefore f is uniformly continuous. □

- Recall that we previously characterized compactness in terms of convergent subnets. On metric spaces we can do this instead with subsequences.

Thm Let (X, d) be a metric space. Then X is compact if and only if every sequence $(x_n)_{n \in \mathbb{N}} \subset X$ has a convergent subsequence.

Proof $(\Longrightarrow)$: Assume X is compact and let $(x_n)_{n \in \mathbb{N}} \subset X$ be a sequence. For each $n \in \mathbb{N}$, let $C_n := \overline{\{x_j \mid j \geq n\}}$. Then

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$$

so $\mathcal{C} := \{C_n \mid n \in \mathbb{N}\}$ is a collection of closed sets with the finite intersection property. By compactness,

$$\bigcap_{n\in\mathbb{N}} C_n \neq \phi$$

so let y be an element of this intersection. Then $y \in \overline{\{x_j \mid j \geq n\}}$ for all $n \in \mathbb{N}$. This means for any $n \in \mathbb{N}$ and any $\varepsilon > 0$, there exists $j \geq n$ satisfying $x_j \in B_d(y, \varepsilon)$. Let $n_1 \in \mathbb{N}$ be such that $n_1 \geq 1$ and $x_{n_1} \in B_d(y, \varepsilon)$. Inductively choose $n_k \in \mathbb{N}$ to be such that $n_k \geq n_{k-1}+1$ and $x_{n_k} \in B_d(y, \frac{1}{k})$. Then $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence converging to y.

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($\Longleftarrow$) Assume every sequence has a convergent subsequence. Let $\mathcal{C}$ be an open cover of X. We require two intermediate claims.

$\boxed{\text{Claim 1}}$ There exists a $\delta > 0$ that whenever $\operatorname{diam}(A) < \delta$ one has $A \subset U$ for some $U \in \mathcal{C}$. (i.e. $\mathcal{C}$ has a Lebesgue number).

We prove this by contradiction. Then for all $n \in \mathbb{N}$, there exists $A_n \subset X$ with $\operatorname{diam}(A) < \frac{1}{n}$ but such that $A_n \not\subset U$ for any $U \in \mathcal{C}$. Let $x_n \in A_n$ and consider the sequence $(x_n)_{n\in\mathbb{N}} \subset X$. By assumption, this has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ say with limit $x \in X$. Since $\mathcal{C}$ covers X, there is some $U \in \mathcal{C}$ with $x \in U$. Since U is open, there exists $\delta > 0$ so that $B_d(x, \delta) \subset U$. Let $n_k \in \mathbb{N}$ be large enough so that $x_{n_k} \in B_d(x, \delta/2)$ and $\frac{1}{n_k} < \delta/2$. Then $\operatorname{diam}(A_{n_k}) < \frac{1}{n_k} < \frac{\delta}{2}$ implies for all $a \in A_{n_k}$

$$d(x,a) \leq d(x, x_{n_k}) + d(x_{n_k}, a) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus $A_{n_k} \subset B_d(x, \delta) \subset U$, a contradiction. $\square$

$\boxed{\text{Claim 2}}$ For all $\varepsilon > 0$, there exists $x_1, \dots, x_n \in X$ so that $X \subset B_d(x_1, \varepsilon) \cup \cdots \cup B_d(x_n, \varepsilon)$. (i.e. X is totally bounded)

We again proceed by contradiction. Assume for some $\varepsilon > 0$, X is not covered by any finite collection of ε-balls. Fix any $x_1 \in X$. Then $X \not\subset B_d(x_1, \varepsilon)$ implies there is some $x_2 \notin B_d(x_1, \varepsilon)$. Then $X \not\subset B_d(x_1, \varepsilon) \cup B_d(x_2, \varepsilon)$ implies there is some $x_3 \notin B_d(x_1, \varepsilon) \cup B_d(x_2, \varepsilon)$. Proceeding inductively, we obtain a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ so that $d(x_n, x_m) \geq \varepsilon$ for all $n, m \in \mathbb{N}$. By assumption, there is a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$, say with limit x. Let $K \in \mathbb{N}$ be such that $d(x, x_{n_k}) < \frac{\varepsilon}{2}$ for all $k \geq K$. But then

$$d(x_{n_k}, x_{n_{k+1}}) \leq d(x_{n_k}, x) + d(x, x_{n_{k+1}}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

contradicting $d(x_n, x_m) \geq \varepsilon$ for all $n, m \in \mathbb{N}$. $\square$

With the claims in hand, we now show $\mathcal{C}$ has a finite subcover. Let $\delta > 0$ be as in Claim 1, and for $\varepsilon := \frac{\delta}{3}$ let $x_1, \dots, x_n \in X$ be as in Claim 2. Observe that for $j = 1, \dots, n$

$$\operatorname{diam}(B_d(x_j, \delta/3)) \leq \frac{2\delta}{3} < \delta$$

Thus $B_d(x_j, \delta/3) \subset U_j$ for some $U_j \in \mathcal{C}$. Then $\{U_1, \dots, U_n\} \subset \mathcal{C}$ is a subcover since

$$X \subset B_d(x_1, \tfrac{\delta}{3}) \cup \cdots \cup B_d(x_n, \tfrac{\delta}{3}) \subset U_1 \cup \cdots \cup U_n.$$

Therefore X is compact. □

- The condition that every sequence has a convergent subsequence is called sequential compactness. For general topological spaces, compactness is not equivalent to sequential compactness.

Ex Let S_Ω be the minimal uncountable well-ordered set. S_Ω is not compact because it has no largest element. However, it is sequentially compact. Let $(x_n)_{n\in\mathbb{N}} \subset S_\Omega$ be a sequence and denote $A := \{x_n \mid n \in \mathbb{N}\}$. If A is finite then there is a constant subsequence and we are done. Otherwise A is infinite, but countable. Thus A has an upper bound $b \in S_\Omega$. Since S_Ω is well-ordered, A has a minimal element a. Hence $A \subset [a,b]$ and $[a,b]$ is compact by the first theorem in §27. We can then proceed exactly as in the ($\Longrightarrow$) direction of the above theorem to show $(x_n)_{n\in\mathbb{N}}$ must have a convergent subsequence. □

- $[0,1]^{[0,1]}$ with the product topology is compact but not sequentially compact, though we will not prove this here.