Exercises:

§22, 23

1. Recall that for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, its norm is $\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$. Consider $X := \mathbf{R}^2 \setminus \{(0,0)\}$ and $S^1 := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\}$ equipped with their subspace topologies, where $\mathbb{R}^2$ has its standard topology.
   (a) Show that $p(\mathbf{x}) := \frac{1}{\|\mathbf{x}\|}\mathbf{x}$ defines a continuous map $p\colon X \to S^1$.
   (b) Show that p is a quotient map.
   (c) Define an equivalence relation $\sim$ on X so that the quotient space $X/\sim$ is homeomorphic to S^1. Give a geometric description of the equivalence classes.

2. Prove whether or each of the following spaces is connected or disconnected.
   (a) $\mathbb{R}$ equipped with the lower limit topology.
   (b) $\mathbb{R}$ equipped with the finite complement topology.
   (c) $\mathbb{R}^{\mathbb{N}}$ equipped with the uniform topology.

3. Let X be a topological space and let $\{Y_j \mid j \in J\}$ be an indexed family of connected subspaces of X. Suppose there exists a connected subspace $Y \subset X$ satisfying $Y \cap Y_j \neq \emptyset$ for all $j \in J$. Show that $Y \cup \bigcup_{j\in J} Y_j$ is connected.

4. Let X and Y be connected spaces and let $A \subsetneq X$ and $B \subsetneq Y$ be proper subsets. Show that $(X \times Y) \setminus (A \times B)$ is connected.

5. Let $p\colon X \to Y$ be a quotient map. Suppose that Y is connected and $p^{-1}(\{y\})$ is connected for every $y \in Y$. Show that X is connected.

6*. Let $C_0 := [0,1] \subset \mathbb{R}$ and for each $n \in \mathbb{N}$ recursively define

$$C_n := C_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right).$$

Then $C := \bigcap_{n=0}^{\infty} C_n$ is called the **Cantor set**. Equip $C \subset \mathbb{R}$ with the subspace topology.
   (a) Show $C = \overline{C} \setminus C^\circ$.
   (b) Show that every $x \in C$ is a limit point of C.
   (c) Show that C is **totally disconnected**: singleton sets are the only connected subsets.

Solutions:

1. (a) First note that $\|p(\mathbf{x})\| = \left\|\frac{1}{\|\mathbf{x}\|}\mathbf{x}\right\| = \frac{1}{\|\mathbf{x}\|}\|\mathbf{x}\| = 1$, so p is indeed valued in S^1.

We will show that p is continuous by showing its coordinate functions are continuous. The coordinate functions are:

$$p_1(x_1,x_2) = \frac{x_1}{\sqrt{x_1^2+x_2^2}} \qquad \text{and} \qquad p_2(x_1,x_2) = \frac{x_2}{\sqrt{x_1^2+x_2^2}}.$$

Observe that $\sqrt{x_1^2+x_2^2} = d((x_1,x_2),(0,0))$ where d is the euclidean metric. Thus this function is continuous by Exercise 3 on Homework 8. Moreover, since d is a metric this function is equal to zero if and only if $(x_1,x_2) = 0$. Hence $\sqrt{x_1^2+x_2^2}$ is continuous and non-zero on X. The functions $(x_1,x_2) \mapsto x_1$ and $(x_1,x_2) \mapsto x_2$ are the coordinate projections and hence continuous. Using a theorem from §21, we see that p_1 and p_2 are continuous as the quotients of the coordinate functions by the non-zero continuous function $\sqrt{x_1^2+x_2^2}$. □

(b) Note that for $\mathbf{x} \in S^1$, $p(\mathbf{x}) = \frac{1}{1}\mathbf{x} = \mathbf{x}$. Thus p is a retraction of X onto S^1 and therefore is a quotient map by Exercise 2.(b) on Homework 9. $\square$

(c) Define an equivalence relation $\sim$ on X by $\mathbf{x} \sim \mathbf{x}'$ iff $p(\mathbf{x}) = p(\mathbf{x}')$. Then a corollary from §22 implies the map $[\mathbf{x}] \mapsto p(\mathbf{x})$ is a homemorphism from $X/\sim$ to S^1. We claim that $[\mathbf{x}] = \{c\mathbf{x} \mid c > 0\}$. Indeed, if $\mathbf{x}' \in [\mathbf{x}]$ then $\mathbf{x}' \sim \mathbf{x}$ and therefore $\frac{1}{\|\mathbf{x}\|}\mathbf{x} = \frac{1}{\|\mathbf{x}'\|}\mathbf{x}'$. This implies $\mathbf{x}' = \frac{\|\mathbf{x}'\|}{\|\mathbf{x}\|}\mathbf{x}$ and $\frac{\|\mathbf{x}'\|}{\|\mathbf{x}\|} > 0$. Conversely, for $c > 0$ we have

$$p(c\mathbf{x}) = \frac{1}{\|c\mathbf{x}\|}c\mathbf{x} = \frac{1}{c\|\mathbf{x}\|}c\mathbf{x} = \frac{1}{\|\mathbf{x}\|}\mathbf{x} = p(\mathbf{x}),$$

and so $c\mathbf{x} \sim \mathbf{x}$ and $c\mathbf{x} \in [\mathbf{x}]$. This proves the claim, and so we see that $[\mathbf{x}]$ consists of the ray starting at the origin (but not including it), passing through $\mathbf{x}$, and then extending off to infinity. $\square$

2. (a) We claim that this space is disconnected with separation $U = (-\infty, 0)$ and $V = [0, \infty)$. Indeed, recall that the half open intervals $[a, b)$ form a basis for the lower limit topology. Thus U and V are open since

$$U = \bigcup_{n=1}^{\infty} [-n, 0) \qquad \text{and } V = \bigcup_{n=1}^{\infty} [0, n).$$

Both sets are clearly non-empty and satisfy $U \cap V = \emptyset$ and $U \cup V = \mathbb{R}$. So U and V do indeed form a separation for $\mathbb{R}$ with this topology. $\square$

(b) We claim that this space is connected. Suppose, towards a contradiction, that $U, V \subset \mathbb{R}$ are a separation for $\mathbb{R}$. Then U and V are non-empty and open, and so $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are both finite. However $U \cap V = \emptyset$ implies $U \subset \mathbb{R} \setminus V$ and thus U is finite. But then $\mathbb{R} = U \cup \mathbb{R} \setminus U$ is a finite union of finite sets, contradiction $\mathbb{R}$ being infinite. Thus no separation of $\mathbb{R}$ exists and therefore $\mathbb{R}$ is connected. $\square$

(c) We claim that this space is disconnected with separation $U, V \subset \mathbb{R}^{\mathbb{N}}$ where U consists of all the bounded sequences and V consists of all the unbounded sequences. These sets are non-empty and satisfy $U \cap V = \emptyset$ and $U \cup V = \mathbb{R}^{\mathbb{N}}$. Thus it suffices to show these sets are open. Let $\mathbf{x} \in U$. Then we claim $B_{\overline{\rho}}(\mathbf{x}, 1) \subset U$, where $\overline{\rho}$ is the uniform metric. Indeed, for $\mathbf{y} \in B_{\overline{\rho}}(\mathbf{x}, 1)$ we have

$$\overline{d}(y_n, x_n) \leq \overline{\rho}(\mathbf{y}, \mathbf{x}) < 1$$

for all $n \in \mathbb{N}$. This implies $|y_n - x_n| = \overline{d}(y_n, x_n) < 1$. Since $\mathbf{x}$ is bounded, there exists $R > 0$ so that $|x_n| \leq R$ for all $n \in \mathbb{N}$, and so

$$|y_n| = |y_n - x_n + x_n| \leq |y_n - x_n| + |x_n| < 1 + R.$$

Thus $\mathbf{y}$ is bounded by $1 + R$ and so $\mathbf{y} \in U$. Since $\mathbf{x} \in U$ was arbitrary, this shows U is open in the uniform topology. Now let $\mathbf{x} \in V$. We again claim $B_{\overline{\rho}}(\mathbf{x}, 1) \subset V$. By the same estimate as above, for $\mathbf{y} \in B_{\overline{\rho}}(\mathbf{x}, 1)$ we have $|y_n - x_n| < 1$ for all $n \in \mathbb{N}$. Since $\mathbf{x}$ is unbounded, for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ with $|x_{n_k}| \geq k$. Using the reverse triangle inequality we have for each $k \in \mathbb{N}$

$$|y_{n_k}| = |x_{n_k} + (y_{n_k} - x_{n_k})| \geq ||x_{n_k}| - |y_{n_k} - x_{n_k}|| \geq |x_{n_k}| - |y_{n_k} - x_{n_k}| \geq k - 1.$$

Thus $\mathbf{y}$ is unbounded and therefore $\mathbf{y} \in V$. Thus V is open and therefore U, V is a separation of $\mathbb{R}^{\mathbb{N}}$. $\square$

3. For each $j \in J$, $Y \cap Y_j \neq \emptyset$ implies $Y \cup Y_j$ is connected. Then since

$$Y \subset \bigcap_{j \in J} (Y \cup Y_j) \neq \emptyset$$

it follows that

$$\bigcup_{j \in J} (Y \cup Y_j) = Y \cup \bigcup_{j \in J} Y_j$$

is connected. $\square$

4. Fix $x_0 \in X \setminus A$, which exists since A is a proper subset. Then $\{x_0\} \times Y$ is connected since it is homeomorphic to Y. Similarly, for any $y \in Y \setminus B$, $X \times \{y\}$ is homeomorphic to X and therefore is connected. Note that
$$\{x_0, y\} \ni (\{x_0\} \times Y) \cap (X \times \{y\}) \neq \emptyset$$
for all $y \in Y \setminus B$. Thus the previous exercise implies
$$(\{x_0\} \times Y) \cup \bigcup_{y \in Y \setminus B} (X \times \{y\}) = (\{x_0\} \times Y) \cup (X \times Y \setminus B)$$
is connected. Denote this set by $Z(x_0)$, and define the set similarly for all $x \in X \setminus A$. We have
$$X \times Y \setminus B \subset \bigcap_{x \in X \setminus A} Z(x) \neq \emptyset,$$
and thus
$$\bigcup_{x \in X \setminus A} Z(x) = \bigcup_{x \in X \setminus A} (\{x\} \times Y) \cup (X \times Y \setminus B) = (X \setminus A \times Y) \cup (X \times Y \setminus B) = (X \times Y) \setminus (A \times B)$$
is connected. □

5. Suppose, towards a contradiction, that $U, V \subset X$ is a separation. We first claim U and V are saturated with respect to p. We know $U \subset p^{-1}(p(U))$ by Exercise 1 on Homework 1. Conversely, for $x \in p^{-1}(p(U))$ we have $y := p(x) \in p(U)$. Thus $p^{-1}(\{y\}) \cap U$ is non-empty. Since $p^{-1}(\{y\})$ is connected and U, V is a separation, we must have $p^{-1}(\{y\}) \subset U$. In particular, $x \in U$. Thus $U = p^{-1}(p(U))$ and so is saturated. Similarly V is identical. Since p is a quotient map, it follows that $p(U)$ and $p(V)$ are open in Y. Moreover, their union is all of Y since $U \cap V = X$ and p is surjective. They are also non-empty since U and V are non-empty and they are disjoint since p is surjective and
$$p^{-1}(p(U) \cap p(V)) = p^{-1}(p(U)) \cap p^{-1}(p(V)) = U \cap V = \emptyset.$$
Hence $p(U), p(V)$ is a separation for Y, contradicting Y being connected. Thus X is connected. □

6*. (a) We first use induction on n to show each C_n is closed. Indeed, $C_0 = [0, 1]$ is closed so suppose we know C_{n-1} is closed for some $n \in \mathbb{N}$. Then
$$C_n = C_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} = C_{n-1} \cap \left[\bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \right]^c.$$
The union of open intervals is open and thus its complement is closed. Therefore C_n is closed since it is the intersection of C_{n-1} and a closed set. Thus all of the C_n are closed by induction. Now, C is therefore closed as the intersection of the closed sets C_n. This implies $C = \overline{C}$. We will obtain the desired equality if we show $C^\circ = \emptyset$. Let $x \in C$. In order to show $x \notin C^\circ$, we must show $(x - \epsilon, x + \epsilon) \notin C$ for any $\epsilon > 0$. Fix some $\epsilon > 0$ and let $n \in \mathbb{N}$ be large enough so that $\frac{1}{3^n} < \epsilon$. We have $x \in C_n$ and let $k = 0, \ldots, 3^{n-1} - 1$ be such that
$$x \in \left[\frac{3k}{3^n}, \frac{1+3k}{3^n} \right] \cup \left[\frac{2+3k}{3^n}, \frac{3+3k}{3^n} \right].$$
The condition $\frac{1}{3^n} < \epsilon$ implies
$$(x - \epsilon, x + \epsilon) \cap \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \neq \emptyset,$$
and thus $(x - \epsilon, x + \epsilon)$ is not contained in C_n. Consequently, $(x - \epsilon, x + \epsilon)$ fails to be contained in C. Since $\epsilon > 0$ was arbitrary we see that $x \notin C^\circ$, and since $x \in C$ was arbitrary we see that $C^\circ = \emptyset$. □

(b) Let $x \in C$. Recall that x is a limit point if and only if $x \in \overline{C \setminus \{x\}}$. Thus it suffices to show for each $\epsilon > 0$ that $(x-\epsilon, x+\epsilon)$ contains some $y \in C$ with $y \neq x$. Fix $\epsilon > 0$ and let $n \in \mathbb{N}$ be large enough so that $\frac{1}{3^n} < \epsilon$. Since $x \in C \subset C_n$, there exists $k = 0, \ldots, 3^{n-1}-1$ so that

$$x \in \left[\frac{3k}{3^n}, \frac{1+3k}{3^n}\right] \cup \left[\frac{2+3k}{3^n}, \frac{3+3k}{3^n}\right].$$

Define $y = \frac{1+3k}{3^n}$ if $x \in \left[\frac{3k}{3^n}, \frac{1+3k}{3^n}\right) \cup \left\{\frac{2+3k}{3^n}\right\}$, and otherwise let $y = \frac{2+3k}{3^n}$. Then $y \in C_n$ and the choice of n then ensures $|x-y| < \epsilon$, or $y \in (x-\epsilon, x+\epsilon)$. We also note that for $d \in \mathbb{N}$

$$\frac{1+3k}{3^n} = \frac{3^d(1+3k)}{3^{n+d}} < \frac{3(3^{d-1}+3^d k)+1}{3^{n+d}}$$

and

$$\frac{1+3k}{3^n} = \frac{3^d(1+3k)}{3^{n+d}} > \frac{3^d(1+3k)-3+2}{3^{n+d}} = \frac{3(3^{d-1}+3^d k-1)+2}{3^{n+d}}$$

This implies $\frac{1+3k}{3^n} \notin \left(\frac{1+3\ell}{3^m}, \frac{2+3\ell}{3^m}\right)$ for any $m > n$ and $\ell = 0, \ldots, 3^{m-1}-1$. Similarly for $\frac{2+3k}{3^n}$. Consequently $y \in C_m$ for all $m > n$ and thus $y \in C$. □

(c) Singleton sets are connected since they cannot have two disjoint non-empty subsets, let alone open ones. Now suppose $A \subset C$ is connected, and suppose, towards a contradiction, that $x, y \in A$ are distinct. Without loss of generality $x < y$. Let $n \in \mathbb{N}$ be large enough so that $\frac{2}{3^n} < |x-y|$, Consequently if $k = 0, \ldots, 3^{n-1}-1$ is such that

$$x \in \left[\frac{3k}{3^n}, \frac{1+3k}{3^n}\right] \cup \left[\frac{2+3k}{3^n}, \frac{3+3k}{3^n}\right],$$

then x and y cannot belong to the same subinterval. In particular, since $x < y$, if x is in the first interval then there exists $z \in \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$ with $x < z < y$, and if x is in the second interval then there exists $z \in \left(\frac{4+3k}{3^n}, \frac{5+3k}{3^n}\right)$ with $x < z < y$. In either case $z \notin C_n$ and thus $z \notin C$. But then $U := (-\infty, z) \cap A$ and $V := (z, \infty) \cap A$ are open in A, non-empty since $x \in U$ and $y \in V$, and satisfy $U \cap V = \emptyset$ and $U \cup V = A$. Thus U, V is a separation for A, which contradicts A being a connected. Thus A must contain at most one element. □