Exercises:

§24, 26

1. Recall that $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
   (a) Show that S^1 is connected.
   (b) Show that $a(x, y) := (-x, -y)$ defines a homeomorphism $a\colon S^1 \to S^1$.
   (c) Show that if $f\colon S^1 \to \mathbb{R}$ is continuous, then there exists $(x, y) \in S^1$ satisfying $f(x, y) = f(-x, -y)$.

2. Let $U \subset \mathbb{R}^n$ be open and connected. Show that U is path connected.
   [**Hint:** for $\mathbf{x}_0 \in U$ show that the set of points $\mathbf{x} \in U$ that are connected to $\mathbf{x}_0$ by a path in U is clopen.]

3. Equip $\mathbb{R}$ with the finite complement topology. Show that every subset is compact.

4. Let X be a Hausdorff space. If $A, B \subset X$ are compact with $A \cap B = \emptyset$, show that there are open sets $U \supset A$ and $V \supset B$ with $U \cap V = \emptyset$.

5. Let $p\colon X \to Y$ be a closed continuous surjective map.
   (a) For $U \subset X$ open, show that $p^{-1}(\{y\}) \subset U$ for $y \in Y$ implies there is a neighborhood V of y with $p^{-1}(V) \subset U$.
   (b) Show that if Y is compact and $p^{-1}(\{y\})$ is compact for each $y \in Y$, then X is compact.

6*. Let G be a topological group with identity $e \in G$.
   (a) For $U \subset G$ a neighborhood of e, show that there exists a neighborhood V of e satisfying $VV \subset U$.
   (b) For $A \subset G$ closed and $B \subset G$ compact with $A \cap B = \emptyset$, show that there exists a neighborhood V of e satisfying $A \cap VB = \emptyset$.
   (c) For $A \subset G$ closed and $B \subset G$ compact, show that AB is closed.
   (d) For $H < G$ a compact subgroup, show that the quotient map $p\colon G \to G/H$ is closed.
   (e) Show that if $H < G$ is a compact subgroup with G/H compact, then G is compact.

Solutions:

1. (a) We have seen in lecture that $f\colon [0, 1] \to S^1$ defined by $f(t) = (\sin(2\pi t), \cos(2\pi t))$ is a continuous bijection. Since $[0, 1]$ is connected, it follows that S^1 is connected. □

   (b) We first show a is valued in S^1. If $(x, y) \in S^1$, then $x^2 + y^2 = 1$. Consequently, $(-x)^2 + (-y)^2 = x^2 + y^2 = 1$ and so $a(x, y) = (-x, -y) \in S^1$. Next, observe that $a(a(x, y)) = a(-x, -y) = (x, y)$. Thus a is its own inverse and hence is bijective. Therefore it suffices to show a is continuous. This follows from the fact that its coordinate functions $a_1(x, y) = -x$ and $a_2(x, y) = -y$ are continuous: they are the coordinate projections, which we know are continuous, times the constant function -1. □

   (c) Define $g\colon S^1 \to \mathbb{R}$ by $g := f - f \circ a$. This since f and $f \circ a$ are both continuous, their difference—g—is continuous. Now take any $\mathbf{x} \in S^1$. If $g(\mathbf{x}) = 0$ then we have $f(\mathbf{x}) = f \circ a(\mathbf{x}) = f(-\mathbf{x})$ and so are done. Otherwise, we have either $g(\mathbf{x}) > 0$ or $g(\mathbf{x}) < 0$. Without loss of generality, assume the former. Observe that, since $a \circ a$ is the identity map, we have

   $$g(-\mathbf{x}) = g \circ a(\mathbf{x}) = f \circ a(\mathbf{x}) - f \circ a \circ a(\mathbf{x}) = f \circ a(\mathbf{x}) - f(\mathbf{x}) = -g(\mathbf{x}).$$

   Thus $g(\mathbf{x}) > 0$ implies $g(-\mathbf{x}) < 0$. Since S^1 is connected by an example from lecture, the intermediate value theorem implies there exists some $\mathbf{y} \in S^1$ satisfying $g(\mathbf{y}) = 0$. Hence $f(\mathbf{y}) = f(-\mathbf{y})$. □

2. As per the hint, we fix $\mathbf{x}_0 \in U$ and let A be the set of all points $\mathbf{x} \in U$ for which there is a path connecting $\mathbf{x}_0$ and $\mathbf{x}$. If $A = U$, then we are done. Indeed, let $\mathbf{x}, \mathbf{y} \in U = A$ so that there exists continuous functions $f_1 \colon [a,b] \to U$ and $f_2 \colon [c,d] \to U$ such that $f_1(a) = \mathbf{x}_0 = f_2(c)$, $f_1(b) = \mathbf{x}$, and $f_2(d) = \mathbf{y}$. Then we can concatenate these two paths to form a path from $\mathbf{x}$ to $\mathbf{y}$ as follows. Define $f \colon [a, b + (d-c)] \to U$ by

$$f(t) = \begin{cases} f_1(a+b-t) & \text{if } a \leq t \leq b \\ f_2(t+c-b) & \text{if } b < t \leq b + (d-c) \end{cases}.$$

Then $f(a) = f_1(b) = \mathbf{x}$ and $f(b+(d-c)) = f_2(b+(d-c)+c-b) = f_2(d) = \mathbf{y}$. Note that $f_1(a+b-t)$ is continuous as the composition of continuous functions: $a+b-t$ and f_1. Similarly, $f_2(t+c-b)$ is continuous. Also note that $f_1(a+b-t)$ and $f_2(t+c-b)$ agree on $[a,b] \cap [b, +b+(d-c)] = \{b\}$ since

$$f_1(a+b-b) = f_1(a) = \mathbf{x}_0 = f_2(c) = f_2(b+c-b).$$

Thus f is continuous by the pasting lemma. Since $\mathbf{x}, \mathbf{y} \in U$ were arbitrary, U is path-connected. Thus it suffices to show $A = U$.

We will show A is clopen in U. Since U is connected, this will imply either $A = \emptyset$ or $A = U$. Note that $A \neq \emptyset$ since $\mathbf{x}_0 \in A$: $\mathbf{x}_0$ is connected to itself via the constant path $f(t) = \mathbf{x}_0$ for all $0 \leq t \leq 1$. Thus if A is clopen we have $A = U$ and so the proof is complete by the above argument.

We first show that A is open in U. Let $\mathbf{x} \in A$. Since U is open, we can find $\epsilon > 0$ such that $B_d(\mathbf{x}, \epsilon) \subset U$, where d is the euclidean metric on $\mathbb{R}^n$. Recall that we showed in lecture that $B_d(\mathbf{x}, \epsilon)$ is path connected. Thus for any $\mathbf{y} \in B_d(\mathbf{x}, \epsilon)$ there is a path from $\mathbf{x}$ to $\mathbf{y}$ contained in $B_d(\mathbf{x}, \epsilon) \subset U$. Concatenating this path with the one from $\mathbf{x}_0$ to $\mathbf{x}$ (which exists since $\mathbf{x} \in A$) as above, we obtain a path from $\mathbf{x}_0$ to $\mathbf{y}$ in U. Hence $\mathbf{y} \in A$ and therefore $B_d(\mathbf{x}, \epsilon) \subset A$. It follows that A is open in U.

Finally we show A is closed in U. Suppose $(\mathbf{x}_n)_{n \in \mathbb{N}} \subset A$ is a sequence which converges to some $\mathbf{x} \in U$. Using that U is open, we can find $\epsilon > 0$ so that $B_d(x, \epsilon) \subset U$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathbf{x}_n \in B_d(\mathbf{x}, \epsilon)$. In particular, $\mathbf{x}_N \in B_d(\mathbf{x}, \epsilon)$. But since $\mathbf{x}_N$ can be connected to $\mathbf{x}_0$ by a path, and $B_d(\mathbf{x}, \epsilon)$ is path-connected, we can as above find a path connecting $\mathbf{x}_0$ and $\mathbf{x}$. Hence $\mathbf{x} \in A$ and so A is closed in U by the sequence lemma. □

3. Fix $A \subset \mathbb{R}$ and let $\mathcal{C}$ be an open cover for A. If A is empty, then $\{U\}$ is a finite subcover for any $U \in \mathcal{U}$. If A is nonempty, let $x \in A$ and let $U \in \mathcal{C}$ be such that $x \in U$. Thus U is nonempty and open, and hence $\mathbb{R} \setminus U$ is finite. Consequently, $A \cap (R \setminus U)$ is finite, and let $\{a_1, \ldots, a_n\}$ be the elements in this intersection. Thus $U \subset A \setminus \{a_1, \ldots, a_n\}$. Since $a_j \in A$ for each $j = 1, \ldots, n$, there exists $U_j \in \mathcal{C}$ with $a_j \in U_j$. Consequently,

$$U \cup U_1 \cup \cdots \cup U_n \supset (A \setminus \{a_1, \ldots, a_n\}) \cup \{a_1\} \cup \cdots \cup \{a_n\} = A.$$

That is, $\{U, U_1, \ldots, U_n\} \subset \mathcal{C}$ is a finite subcover. Since $\mathcal{C}$ was arbitrary, we see that A is compact. □

4. By a lemma from lecture, for each $b \in B$ there exists disjoint open sets $U_b, V_b \subset X$ with $A \subset U_b$ and $b \in V_b$. Observe that $\{V_b \mid b \in B\}$ is an open cover of B, so compactness yields a finite subcover $\{V_{b_1}, \ldots, V_{b_n}\}$. Define

$$U := \bigcap_{j=1}^{n} U_{b_j} \qquad \text{and} \qquad V := V_{b_1} \cup \cdots \cup V_{b_n}.$$

Then U is open since it is the *finite* intersection of open sets, and V is open since it is the union of open sets. Since $A \subset U_{b_j}$ for each $j = 1, \ldots, n$ we have $A \subset U$. We also have $B \subset V$ since $\{V_{b_1}, \ldots, V_{b_n}\}$ is a subcover. Finally,

$$U \cap V = (U \cap V_{b_1}) \cup \cdots \cup (U \cap V_{b_n}) \subset (U_{b_1} \cap V_{b_1}) \cup \cdots \cup (U_{b_n} \cap V_{b_n}) = \emptyset$$

□

5. (a) Suppose $U \subset X$ is open with $p^{-1}(\{y\}) \subset U$ for $y \in Y$. Then $X \setminus U$ is closed and since p is a closed map we have that $p(X \setminus U)$ is closed in Y. Thus $V := Y \setminus p(X \setminus U)$ is open. Note that $y \in V$, since otherwise $y \in p(X \setminus U)$ and so there exists $x \in X \setminus U$ with $p(x) = y$, but this contradicts $p^{-1}(\{y\}) \subset U$. Thus V is a neighborhood of y. If $x \in p^{-1}(V)$, then $p(x) \notin p(X \setminus U)$. Thus we must have $x \notin X \setminus U$ and therefore $x \in U$. That is, $p^{-1}(V) \subset U$. □

(b) Let $\mathcal{C}$ be an open cover for X. Then $\mathcal{C}$ is an open cover for $p^{-1}(\{y\})$ for each $y \in Y$. The compactness of $p^{-1}(\{y\})$ implies there is a finite subcover $\mathcal{S}_y \subset \mathcal{C}$. Using part (a), there exists a neighborhood V_y of y with

$$p^{-1}(V_y) \subset \bigcup_{U \in \mathcal{S}_y} U.$$

Now, $\{V_y \mid y \in Y\}$ is an open cover of Y. Since Y is compact, there is a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Note that

$$\mathcal{S} := \bigcup_{j=1}^{n} \mathcal{S}_{y_j} \subset \mathcal{C}$$

is finite since each $\mathcal{S}_{y_j}$ is finite. Also we have

$$X = p^{-1}(Y) \subset p^{-1}\left(\bigcup_{j=1}^{n} V_{y_j}\right) = \bigcup_{j=1}^{n} p^{-1}(V_{y_j}) \subset \bigcup_{j=1}^{n} \bigcup_{U \in \mathcal{S}_{y_j}} U = \bigcup_{U \in \mathcal{S}} U.$$

Thus $\mathcal{S} \subset \mathcal{C}$ is a finite subcover and X is compact. □

6*. (a) Let $m\colon G \times G \to G$ be the multiplication map. Since this is continuous, $m^{-1}(U)$ is a neighborhood of $(e, e) \in G \times G$. Since cartesian products of open sets form a basis for the topology on $G \times G$, there exists open subsets $V_1, V_2 \subset G$ with $(e, e) \in V_1 \times V_2 \subset m^{-1}(U)$. If we let $V := V_1 \cap V_2$, then V is a neighborhood of e and

$$VV = m(V \times V) \subset m(V_1 \times V_2) \subset m(m^{-1}(U)) \subset U.$$

□

(b) Let $x \in B$ so that $x \in G \setminus A$. Since $G \setminus A$ is open, $(G \setminus A)x^{-1}$ is a neighborhood of e. By part (a), there exists V_x a neighborhood of e satisfying $V_x V_x \subset (G \setminus A)x^{-1}$. Then $V_x x$ is a neighborhood of x and thus $\{V_x x \mid x \in B\}$ is an open cover of B. By compactness we have a finite subcover $\{V_{x_1} x_1, \ldots, V_{x_n} x_n\}$. Define

$$V := \bigcap_{j=1}^{n} V_{x_j},$$

which is a neighborhood of e. For each $x \in B$ we have $x \in V_{x_k} x_k$ for some $k \in \{1, \ldots, n\}$. Thus

$$Vx \subset V_{x_k} x \subset V_{x_k} V_{x_k} x_k \subset (G \setminus A) x_k^{-1} x_k = G \setminus A.$$

Since $x \in B$ was arbitrary, we have $VB \subset G \setminus A$, or $A \cap VG = \emptyset$. □

(c) We will show the complement of AB is open. Let $x \notin AB$. Since taking inverses is continuous, B^{-1} is compact. Also xB^{-1} is compact since multiplying by x is continuous. Now $A \cap xB^{-1} = \emptyset$ since otherwise $xb^{-1} = a$ or $x = ab$ for some $a \in A$ and $b \in B$, contradicting $x \notin AB$. So A is a closed subset disjoint from the compact subset xB^{-1}. The previous part implies there is an open neighborhood V of e such that $A \cap VxB^{-1} = \emptyset$. This is equivalent to $AB \cap Vx = \emptyset$, or $Vx \subset G \setminus (AB)$. Since V is a neighborhood of e, Vx is a neighborhood of x. Thus the complement of AB is open and therefore AB is closed. □

(d) Let $A \subset G$ be closed. Then by part (c), AH is closed. We also note that AH is saturated: if $x \in p^{-1}(p(AH))$ then $xH \in p(AH) = \{yH \mid y \in AH\}$. Thus $xH = yH$ for some $y \in \mathcal{H}$, which implies $x = yh$ for some $h \in \mathcal{H}$. Since $y \in AH$ and H is a subgroup, it follows that $x \in AH$. Thus $p^{-1}(p(AH)) = AH$. Since AH is saturated and p is the quotient map, $p(AH)$ is closed. But $p(A) = p(AH)$ and so p is closed. □

(e) By part (d), $p\colon G \to G/H$ is closed. It is also continuous and surjective. Note that $p^{-1}(\{gH\}) = gH$, which is compact since H is compact. Since G/H is also assumed to be compact, Exercise 5 implies G is compact. $\square$