Exercises: (§2, 3, 6, 7)

1. Let $f\colon A \to B$ be a function.
   (a) For $A_0 \subset A$ and $B_0 \subset B$, show that $A_0 \subset f^{-1}(f(A_0))$ and $f(f^{-1}(B_0)) \subset B_0$.
   (b) Show that f is injective if and only if $A_0 = f^{-1}(f(A_0))$ for all subsets $A_0 \subset A$.
   (c) Show that f is surjective if and only if $f(f^{-1}(B_0)) = B_0$ for all subsets $B_0 \subset B$.

2. Let C be a relation on a set A. For a subset $A_0 \subset A$, the **restriction** of C to A_0 is the relation defined by the subset $D := C \cap (A_0 \times A_0)$.
   (a) For $a, b \in A$, show that aDb if and only if $a, b \in A_0$ and aCb.
   (b) Show that if C is an equivalence relation on A, then D is an equivalence relation on A_0.
   (c) Show that if C is an order relation on A, then D is an order relation on A_0.
   (d) Show that if C is a partial order relation on A, then D is a partial order relation on A_0.

3. Let A and B be non-empty sets.
   (a) Prove that $A \times B$ is finite if and only if A and B are both finite.
   (b) Let B^A denote the set of functions $f\colon A \to B$. Show that if A and B are finite, then so is B^A.
   (c) Suppose B^A is finite and B has at least two elements. Show that A and B are finite.

4. We say two sets A and B have the same **cardinality** if there is a bijection of A with B. In this exercise, you will prove the *Schröder–Bernstein Theorem*: if there exist injections $f\colon A \to B$ and $g\colon B \to A$, then A and B have the same cardinality.
   (a) Suppose $C \subset A$ and that there is an injection $f\colon A \to C$. Define $A_1 := A$, $C_1 := C$, and for $n > 1$ recursively define $A_n := f(A_{n-1})$ and $C_n := f(C_{n-1})$. Show that
   $$A_1 \supset C_1 \supset A_2 \supset C_2 \supset A_3 \supset \cdots$$
   and that $f(A_n \setminus C_n) = A_{n+1} \setminus C_{n+1}$ for all $n \in \mathbb{N}$.
   (b) Using the notation from the previous part, show that $h\colon A \to C$ defined by
   $$h(x) := \begin{cases} f(x) & \text{if } x \in A_n \setminus C_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$
   is a bijection. [**Hint:** draw a picture.]
   (c) Prove the Schröder–Bernstein Theorem.

5. Let $\{0,1\}^{\mathbb{N}}$ denote the set of functions $f\colon \mathbb{N} \to \{0,1\}$.
   (a) Show that $\{0,1\}^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality.
   (b) Let $\mathcal{C}$ be the collection of *countable* subsets of $\{0,1\}^{\mathbb{N}}$. Show that $\mathcal{C}$ and $\{0,1\}^{\mathbb{N}}$ have the same cardinality. [**Hint:** first construct an injection from $\mathcal{C}$ to $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ then use Exercise 4.]

Solutions:

1. (a) Let $a \in A_0$. Then $f(a) \in f(A_0)$ and therefore $a \in f^{-1}(f(A_0))$. Since $a \in A_0$ was arbitrary, we have $A_0 \subset f^{-1}(f(A_0))$. Next, let $b \in f(f^{-1}(B_0))$. Then there exists $a \in f^{-1}(B_0)$ such that $f(a) = b$. But $a \in f^{-1}(B_0)$ implies $b = f(a) \in B_0$. Since $b \in f(f^{-1}(B_0))$ was arbitrary, we have $f(f^{-1}(B_0)) \subset B_0$. □

(b) $(\Longrightarrow)$: Suppose f is injective and let $A_0 \subset A$. By the previous part, it suffices to show $f^{-1}(f(A_0)) \subset A_0$. If $a \in f^{-1}(f(A_0))$, then $f(x) \in f(A_0)$ and so there is some $a_1 \in A_0$ with $f(a) = f(a_1)$. Since f is injective, we must have $a = a_1 \in A_0$. Thus $f^{-1}(f(A_0)) \subset A_0$.

$(\Longleftarrow)$: We will proceed by contrapositive. Suppose f is **not** injective. Then there exists $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$. Consider $A_0 := \{a_1\}$. Then $f(\{a_1\}) = \{f(a_1)\}$ and so $a_1, a_2 \in f^{-1}(f(\{a_1\}))$. Consequently, $\{a_1\}$ does **not** equal $f^{-1}(f(\{a_1\}))$ (it is a strict subset). □

(c) $(\Longrightarrow)$: Suppose f is surjective and let $B_0 \subset B$. By part (a) it suffices to show $B_0 \subset f(f^{-1}(B_0))$. Let $b \in B_0$. Since f is surjective, we can find some $a \in A$ with $f(a) = b$. Consequently, $a \in f^{-1}(B_0)$ and $b = f(a) \in f(f^{-1}(B_0))$. Thus $B_0 \subset f(f^{-1}(B_0))$.

$(\Longleftarrow)$: We will again proceed by contrapositive. Suppose f is **not** surjective. Then there exists $b \in B$ so that $f(a) \neq b$ for all $a \in A$. Consider $B_0 := \{b\}$. Since nothing in A is mapped to b by f, we have $f^{-1}(\{b\}) = \emptyset$. Thus $f(f^{-1}(\{b\})) = \emptyset \neq \{b\}$. □

2. (a) If aDb, then this means $(a, b) \in D = C \cap (A_0 \times A_0)$. In particular, $(a, b) \in C$ so that aCb, and $(a, b) \in A_0 \times A_0$ so that $a, b \in A_0$. Conversely, if $a, b \in A_0$ and aCb, then the former implies $(a, b) \in A_0 \times A_0$ and the latter implies $(a, b) \in C$. Thus (a, b) is in their intersection, which is D, and consequently aDb. □

(b) Let C be an equivalence relation on A and let D be its restriction to a subset $A_0 \subset A$. So C satisfies reflexivity, symmetry, and transitivity and we must show D inherits these properties. For $a \in A_0$, we have aCa by reflexivity and consequently aDa by part (a). Thus D is reflexive. For $a, b \in A_0$, if aDb, then aCb by part (a). By symmetry of C we have bCa and since we still have $a, b \in A_0$, we obtain bDa by part (a). Thus D is symmetric. Finally, for $a, b, c \in A_0$, if aDb and bDc, then we have aCb and bCc, and so aCc by transitivity of C. Using part (a) again we obtain aDc whence D is transitive. □

(c) Let C be an order relation on A and let D be its restriction to a subset $A_0 \subset A$. So C satisfies comparability, non-reflexivity, and transitivity and we must show D inherits these properties. Let $a, b \in A_0$ with $a \neq b$. Then aCb by comparability, and consequently aDb by part (a); that is, D has comparability. Let $a \in A_0$. If aDa, then aCa by part (a), which contradicts non-reflexivity of C. Thus aDa holds for no $a \in A_0$, which means D has non-reflexivity. Finally, the proof of transitivity follows by exactly the same argument as in part (b). □

(d) Let C be a partial order relation on A and let D be its restriction to a subset $A_0 \subset A$. So C satisfies reflexivity, antisymmetry, and transitivity and we must show D inherits these properties. Reflexivity and transitivity follows by the same arguments as in part (b), so it suffices show D is antisymmetric. If $a, b \in A_0$ satisfy aDb and bDa, then we have aCb and bCa by part (a). Since C is antisymmetric, we must have $a = b$. Thus D is antisymmetric. □

3. (a) $(\Longrightarrow)$: Suppose $A \times B$ is finite. Then by Corollary 6.7, there is an injective function $f \colon A \times B \to \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Let $a_0 \in A$ and $b_0 \in B$ (which exist since A and B are assumed to be non-empty), and note that the maps

$$\iota_A \colon A \ni a \mapsto (a, b_0) \in A \times B$$
$$\iota_B \colon B \ni b \mapsto (a_0, b) \in A \times B$$

are injective. Consequently, $f \circ \iota_A \colon A \to \{1, 2, \ldots, 2\}$ and $f \circ \iota_B \colon B \to \{1, 2, \ldots, n\}$ are injective maps as compositions of injective maps. Thus A and B are finite by Corollary 6.7.

$(\Longleftarrow)$: Suppose A and B are finite. By Corollary 6.7, there are injective functions $f \colon A \to \{1, 2, \ldots, n\}$ and $g \colon B \to \{1, 2, \ldots, m\}$. Note that $n, m \geq 1$ since A and B are both non-empty. Observe that the map

$$h \colon \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} \to \{1, 2, \ldots, nm\}$$
$$(i, j) \mapsto (i - 1)m + j$$

is injective. Indeed, if $h(i, j) = h(i', j')$ then $(i - i')m = j' - j$, which implies $j' - j$ is divisible by m. Since $j' - j \in \{-m + 1, -m + 2, \ldots, -1, 0, 1, \ldots, m - 2, m - 1\}$, this is only possible if

$j' - j = 0$ in which case $(i - i')m = 0$. Thus $j = j'$ and $i = i'$ and h is injective. Consider the map $k\colon A \times B \to \{1, 2, \ldots, nm\}$ defined by $k(a, b) := h(f(a), g(b))$. We claim this is injective, in which case $A \times B$ is finite by Corollary 6.7. Suppose $k(a, b) = k(a', b')$. Then $h(f(a), g(b)) = h(f(a'), g(b'))$. Since h is injective, we must have $(f(a), g(b)) = (f(a'), g(b'))$. So $f(a) = f(a')$ and $g(b) = g(b')$, but since each of these functions is injective we obtain $a = a'$ and $b = b'$. Thus k is injective. $\square$

(b) Let n be the cardinality of A and m the cardinality of B. We will show that there is a bijection between B^A and B^n, and then use the previous part (and induction) to show B^n is finite. Since A has cardinality n, there is a bijection $\sigma\colon \{1, 2, \ldots, n\} \to A$. So we can define a map $\phi\colon B^A \to B^n$ by $\phi(f) := (f(\sigma(1)), \ldots, f(\sigma(n)))$ for $f \in B^A$. Suppose $\phi(f) = \phi(f')$ for $f, f' \in B^A$. Then $f(\sigma(j)) = f'(\sigma(j))$ for each $j = 1, \ldots, n$. This implies $f = f'$ because each $a \in A$ occurs in the set $\{\sigma(1), \ldots, \sigma(n)\}$. Thus ϕ is injective. Also, given any $(b_1, \ldots, b_n) \in B^n$ the function $f \in B^A$ defined by $f(a) := b_{\sigma^{-1}(a)}$ satisfies $\phi(f) = (b_1, \ldots, b_n)$. Thus ϕ is also surjective. So it now suffices to show B^n is finite, and we will proceed by induction on n. If $n = 1$, then this is immediate from the finiteness of B. So suppose we know B^{n-1} is finite. Then $B^n = B^{n-1} \times B$, and consequently B^n is finite by part (a). Induction then concludes the proof. $\square$

(c) We will first show A is finite. Let $b_1, b_2 \in B$ be distinct elements. For a fixed $a \in A$, define $f_a\colon A \to B$ by $f_a(a) = b_1$ and $f_a(a') = b_2$ for $a' \neq a$.Then $a \mapsto f_a$ is an injection from A into B^A. Since B^A is finite, there is an injection from B^A to $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$. The composition of these injections, along with Corollary 6.7 shows A is finite. Next, we show B is finite. For each $b \in B$, define $g_b\colon A \to B$ by $g_b(a) := b$ for all $a \in B$. Then $b \mapsto g_b$ is an injection from B to B^A. By the same argument as with A, this implies B is finite. $\square$

4. (a) We will establish this series of containments by proving "$A_n \supset C_n \supset A_{n+1}$" via induction on n. For $n = 1$, we have $A_1 = A$, $C_1 = C$, and $A_2 = f(A)$. So the inclusion $A_1 \supset C_1$ follows from the fact that C is a subset of A, and the inclusion $C_1 \supset A_2$ follows from the fact that the C is the range of f. Now assume $A_{n-1} \supset C_{n-1} \supset A_n$. Then appllying f yields $f(A_{n-1}) \supset f(C_{n-1}) \supset f(A_n)$, but this is precisely the series of inclusions $A_n \supset C_n \supset A_{n+1}$. Thus the full series of inclusions holds by induction.

Now, we must show $f(A_n \setminus C_n) = A_{n+1} \setminus C_{n+1}$ for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and let $a \in A_n \setminus C_n$. Then $f(a) \in A_{n+1}$ by definition of A_{n+1}. We also cannot have $f(a) \in C_{n+1}$ because $C_{n+1} = f(C_n)$ would imply that $f(a) = f(c)$ for some $c \in C_n$ and hence $a = c \in C_n$ since f is injective, a contradiction. Thus $f(a) \in A_{n+1} \setminus C_{n+1}$, and so $f(A_n \setminus C_n) \subset A_{n+1} \setminus C_{n+1}$. Conversely, let $b \in A_{n+1} \setminus C_{n+1}$. Then $A_{n+1} = f(A_n)$ implies there is some $a \in A_n$ with $f(a) = b$. We must also have $a \notin C_n$ because otherwise $b = f(a) \in C_{n+1}$, a contradiction. Thus $A_{n+1} \setminus C_{n+1} \subset f(A_n \setminus C_n)$ and so the sets are equal.

(b) We first show h is injective. Suppose $h(x) = h(y)$. If $x \in A_n \setminus C_n$ for some $n \in \mathbb{N}$, then $h(y) = h(x) = f(x) \in A_{n+1} \setminus C_{n+1}$ by part (a). We cannot have $h(y) = y$ because this would require (by definition of h) that $y \notin A_n \setminus C_n$ for any n, and yet $y = h(y) = f(x) \in A_{n+1} \setminus C_{n+1}$. Thus we must have $h(y) = f(y)$, and so $f(y) = f(x)$. Since f is injective, this implies $x = y$. If $x \notin A_n \setminus C_n$ for all $n \in \mathbb{N}$, then $h(x) = x$ by definition of h. By the same reasoning as above, we cannot have $y \in A_m \setminus C_m$ for any m, and so we have $y = h(y) = h(x) = x$. Thus h is injective.

Next we show h is surjective. Let $y \in C$. If $y \notin A_n \setminus C_n$ for any $n \in \mathbb{N}$, then $h(y) = y$ and so y is in the image of h. If $y \in A_n \setminus C_n$ for some n, then we must have $n > 1$ since $y \in C = C_1$. Thus, by part (a), $A_n \setminus C_n = f(A_{n-1} \setminus C_{n-1})$. So there is some $x \in A_{n-1} \setminus C_{n_1}$ with $f(x) = y$. Since $x \in A_{n-1} \setminus C_{n-1}$, we have $h(x) = f(x) = y$. Thus h is surjective.

(c) Suppose $f\colon A \to B$ and $g\colon B \to A$ are injections. Consider $C := g(B) \subset A$ and note that $g \circ f\colon A \to C$ is an injection. So by part (b), there is a bijection $h\colon A \to C$. Since g is an injection, by changing the range of g we get that $g\colon B \to g(B) = C$ is a bijection. Hence $g^{-1} \circ h\colon A \to B$ is a bijection and so A and B have the same cardinality. $\square$

5. (a) Given $f \in \{0, 1\}^{\mathbb{N}}$, define a subset a subset of the natural numbers by $A_f := \{n \in \mathbb{N} \mid f(n) = 1\}$. We claim that $f \mapsto A_f$ is a bijection $\{0, 1\}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$. If $A_f = A_{f'}$, then for each $n \in \mathbb{N}$ we have

$f(n) = f'(n) = 1$ if $n \in A_f = A_{f'}$ and $f(n) = f'(n) = 0$ otherwise. Thus $f \mapsto A_f$ is injective. Given $A \in \mathcal{P}(\mathbb{N})$, define $f\colon \mathbb{N} \to \{0,1\}$ by $f(n) = 1$ if $n \in A$ and $f(n) = 0$ otherwise. Then $A_f = A$ and so the map is also surjective. Thus $\{0,1\}^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality. □

(b) We will show there are injections $\mathcal{C} \to \{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}} \to \mathcal{C}$ and then use the Schröder–Bernstein Theorem. The latter is easy to define: simply send $f \in \{0,1\}^{\mathbb{N}}$ to $\{f\} \in \mathcal{C}$. For the the former, we will actually define intermediate injections $\mathcal{C} \to (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$.

If $C \in \mathcal{C}$, then by Theorem 7.1 there is a surjective function $f_C\colon \mathbb{N} \to C$. Changing the range of f_C to all of $\{0,1\}^{\mathbb{N}}$, we can view $f_C \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ where C is the image of f. Then for $C, C' \in \mathcal{C}$, if $f_C = f_{C'}$ then in particular the image of f_C (which is C) equals the image of $f_{C'}$ (which is C'). Thus $C \mapsto f_C$ is an injection $\mathcal{C} \to (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$. It remains to show there is an injection $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$. First recall that since $\mathbb{N} \times \mathbb{N}$ is countably infinite, there is a bijection $g\colon \mathbb{N} \to \mathbb{N} \to \mathbb{N}$. Now, given $f \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, we view it as a function $f\colon \mathbb{N} \to \{0,1\}^{\mathbb{N}}$. That is, for each $n \in \mathbb{N}$, $f(n) \in \{0,1\}^{\mathbb{N}}$ and so $f(n)\colon \mathbb{N} \to \{0,1\}$. Thus $(f(n))(m) \in \{0,1\}$ for each $n, m \in \mathbb{N}$, which means we can view f as a function $f\colon \mathbb{N} \times \mathbb{N} \to \{0,1\}$. Consequently, $f \circ g\colon \mathbb{N} \to \{0,1\}$, or $f \circ g \in \{0,1\}^{\mathbb{N}}$. We claim $f \mapsto f \circ g$ is the desired injection. Indeed, if $f \circ g = f' \circ g$ for $f, f' \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, then for any $(n,m) \in \mathbb{N} \times \mathbb{N}$ let $k = g^{-1}(n,)$. We have $f(n,m) = f(g(k)) = f'(g(k)) = f'(n,m)$. Since $(n,m) \in \mathbb{N} \times \mathbb{N}$ was arbitrary, we obtain $f = f'$ and so $f \mapsto f \circ g$ is injective. □