Exercises:

§13, 14, 15, 16

1. Equip $\mathbb{R}$ with the standard topology. Show that a set $U \subset \mathbb{R}$ is open if and only if for all $x \in U$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$.

2. Let X be a space.
   (a) Let $\{\mathcal{T}_i \mid i \in I\}$ be a non-empty collection topologies on X (indexed by some set I). Show that $\bigcap_{i \in I} \mathcal{T}_i$ is a topology on X.
   (b) Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Show that $\mathcal{T}$ is the intersection of all topologies on X that contain $\mathcal{B}$.
   (c) Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on a space X. Suppose $\mathcal{T}'$ is another topology on X that contains $\mathcal{S}$. Show that $\mathcal{T}$ is coarser than $\mathcal{T}'$.
   (d) Let $\mathcal{S}$ and $\mathcal{T}$ be as in the previous part. Show that $\mathcal{T}$ is the intersection of all topologies on X that contain $\mathcal{S}$.

3. Let X be an ordered set (with at least two elements) equipped with the order topology. For a subspace $Y \subset X$, show that the collection $\mathcal{S}$ consisting of sets of the form $Y \cap (-\infty, a)$ or $Y \cap (a, +\infty)$ for $a \in X$ form a subbasis for the subspace topology on Y.

4. Let X and Y be topological spaces. A function $f\colon X \to Y$ is called an **open map** if for every open subset $U \subset X$ one has that its image $f(U)$ is open in Y.
   (a) Equip $X \times Y$ with the product topology. Show that the coordinate projections $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are open maps.
   (b) Let $\mathcal{B}$ be a basis for the topology on X and suppose $f(B)$ is open for all $B \in \mathcal{B}$. Show that f is an open map.
   (c) Show that the previous part does not hold for subbases. [**Hint:** consider the function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 1$ and $f(x) = |x|$ if $x \neq 0$ where $\mathbb{R}$ has the standard topology.]

5. Equip $\mathbb{R}$ with the standard topology.
   (a) Show that the subspace topology on $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is the discrete topology.
   (b) Show that the subspace topology on $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is **not** the discrete topology.

6. In this exercise, you will show that there is a *countable* basis that generates the standard topology on $\mathbb{R}$. For parts (a)–(c), you should only use the properties of $\mathbb{Z}$ and $\mathbb{R}$ given in §4.
   (a) For $x \in \mathbb{R}$, show that there is exactly one $n \in \mathbb{Z}$ satisfying $n \leq x < n + 1$.
   (b) For $x, y \in \mathbb{R}$, show that if $x - y > 1$ then there is at least one $n \in \mathbb{Z}$ satisfying $y < n < x$.
   (c) For $x, y \in \mathbb{R}$, show that if $x - y > 0$ then there exists $z \in \mathbb{Q}$ satisfying $y < z < x$.
   (d) Let $\mathcal{B}$ be the collection of open intervals $(a, b) \subset \mathbb{R}$ with $a, b \in \mathbb{Q}$. Show that $\mathcal{B}$ is countable and is a basis for a topology on $\mathbb{R}$.
   (e) Show $\mathcal{B}$ generates the standard topology on $\mathbb{R}$.

Solutions:

1. Recall that the standard topology is the topology generated by the basis of open intervals. Suppose $U \subset \mathbb{R}$ is open. Then for $x \in U$ there exists an open interval (a, b) satisfying $x \in (a, b) \subset U$. Now, $x \in (a, b)$ implies $a < x < b$ and in particular $\epsilon := \min b - x, x - a > 0$. It follows that $(x - \epsilon, x + \epsilon) \subset (a, b) \subset U$. Conversely, suppose $U \subset \mathbb{R}$ satisfies that for all $x \in U$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Since $(x - \epsilon, x + \epsilon)$ is an open interval (hence a basis set) and contains x, we see that U is open in the topology generated by the open sets; that is, U is open in the standard topology. $\square$

2. (a) We will verify for $\mathcal{T}_0 := \bigcap_{i\in I} \mathcal{T}_i$ the three conditions in the definition of a topology. First, we have $\emptyset, X \in \mathcal{T}_i$ for all $i \in I$ and thus $\emptyset, X \in \mathcal{T}_0$. Next, if $\mathcal{S} \subset \mathcal{T}_0$ is a subcollection, then this same subcollection is containend in every $\mathcal{T}_i$, $i \in I$. As each of these is a topology, we have

$$\bigcup_{U\in\mathcal{S}} U \in \mathcal{T}_i$$

for each $i \in I$. Consequently this union belongs to $\mathcal{T}_0$. Lastly, if $U_1, \ldots, U_n \in \mathcal{T}_0$, then these sets also belong to $\mathcal{T}_i$ for every $i \in I$ whence $U_1 \cap \cdots \cap U_n \in \mathcal{T}_i$ for every $i \in I$. Therefore $U_1 \cap \cdots \cap U_n \in \mathcal{T}_0$ and so $\mathcal{T}_0$ is a topology. □

(b) Let $\mathcal{T}'$ be the intersection of all topologies containing $\mathcal{B}$. This collection of topologies is non-empty (it contains $\mathcal{T}$ for example) and so by the previous part it is a topology on X. Because $\mathcal{T}$ is one the topologies in this collection (by virtue of containing $\mathcal{B}$), we immediately obtain $\mathcal{T}' \subset \mathcal{T}$. On the other hand, observe that $\mathcal{B} \subset \mathcal{T}'$ and we showed in §13 that any topology containing $\mathcal{B}$ contains $\mathcal{T}$. Thus $\mathcal{T} \subset \mathcal{T}'$ and so $\mathcal{T} = \mathcal{T}'$. □

(c) If $\mathcal{T}'$ contains $\mathcal{S}$ and is a topology, then $\mathcal{T}'$ contains all finite intersections of sets from $\mathcal{S}$, and all unions of such intersection. But we showed §13 that $\mathcal{T}$ consists of precisely such sets, and so $\mathcal{T} \subset \mathcal{T}'$. □

(d) Let $\mathcal{T}'$ be the intersection of all topologies containing $\mathcal{S}$. This collection of topologies is non-empty (it contains $\mathcal{T}$ for example) and so by part (a) it is a topology on X. Because $\mathcal{T}$ is one the topologies in this collection (by virtue of containing $\mathcal{S}$), we immediately obtain $\mathcal{T}' \subset \mathcal{T}$. On the other hand, $\mathcal{S} \subset \mathcal{T}'$ and so by the previous part we have $\mathcal{T}' \subset \mathcal{T}$. Thus $\mathcal{T} = \mathcal{T}'$. □

3. We first show this collection $\mathcal{S}$ is a subbasis (i.e. that its union is all of Y). Since X contains at least two elements, we can find $a, b \in X$ with $a < b$. Then for all $x \in X$ we have either $x < a$, $x = a$, or $a < x$. In the first two cases we have $x < b$ and so $x \in (-\infty, b)$, while in the last case we have $x \in (a, +\infty)$. Since $x \in X$ was arbitrary, we have shown

$$X = (a, +\infty) \cup (-\infty, b).$$

Consequently,

$$Y = Y \cap [(a, +\infty) \cup (-\infty, b)] = [Y \cap (a, +\infty)] \cup [Y \cap (b, +\infty)].$$

The latter set is contained in the union over all of $\mathcal{S}$, and so $\mathcal{S}$ is indeed a subbasis for Y.

Let $\mathcal{T}$ be the topology on Y generated by $\mathcal{S}$, and let $\mathcal{T}'$ be the subspace topology on Y. Since open rays are open in X, the sets in $\mathcal{S}$ are open in the subspace topology on Y. Thus $\mathcal{S} \subset \mathcal{T}'$. By Exercise 2.(c), we obtain $\mathcal{T} \subset \mathcal{T}'$. Conversely, that open rays are a subbasis for the order topology on X. Consequently the collection $\mathcal{B}$ of finite intersections of open rays forms a basis for the order topology on X. We saw in §16 that this implies

$$\mathcal{B}_Y := \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology $\mathcal{T}'$ on Y. Observe that each $Y \cap B \in \mathcal{B}_Y$ is a finite intersection of sets from $\mathcal{S}$ and therefore $Y \cap B \in \mathcal{T}$. Thus $\mathcal{B}_Y \subset \mathcal{T}$, which implies $\mathcal{T}' \subset \mathcal{T}$. Hence $\mathcal{T} = \mathcal{T}'$. □

4. For parts (a) and (b) we will require the following claim: the image of a union of sets is the union of images. Indeed, if $f\colon A \to B$ is a function and $\mathcal{C}$ is a collection of subsets $C \subset A$, then for $b \in f\left(\bigcup_{C\in\mathcal{C}} C\right)$ we have $b = f(a)$ for some $a \in \bigcup_{C\in\mathcal{C}} C$. Thus $c \in C$ for some $C \in \mathcal{C}$ and so

$$b = f(a) \in f(C) \subset \bigcup_{C\in\mathcal{C}} f(C).$$

Conversely, if $b \in \bigcup_{C\in\mathcal{C}} f(C)$, then $b \in f(C)$ for some $C \in \mathcal{C}$ and hence $b = f(a)$ for some $a \in C$. Since $a \in C \subset \bigcup_{C\in\mathcal{C}} C$, we have

$$b = f(a) \in f\left(\bigcup_{C\in\mathcal{C}} C\right).$$

This prove the claim.

(a) Recall that the basis for the product topology on $X \times Y$ is the collection of sets of the form $U \times V$ for open subsets $U \subset X$ and $V \subset Y$. Thus an arbitrary open set in $X \times Y$ is of the form $W := \bigcup_{i\in I} U_i \times V_i$ for some indexing set I and open subsets $U_i \subset X$ and $V_i \subset Y$. From the above claim, we have

$$\pi_1(W) = \bigcup_{i\in I} \pi_1(U_i \times V_i) = \bigcup_{i\in I} U_i,$$

which is open in X as the union of open subsets. Similarly, $\pi_2(W) = \bigcup_{i\in I} V_i$, which is open in Y. Thus π_1 and $\pi 2_2$ are open maps. □

(b) Let $\mathcal{T}$ be the topology on X. Then for any $U \in \mathcal{T}$ we have

$$U = \bigcup_{\mathcal{B}\ni B\subset U} B.$$

Thus, using the above claim, we have

$$f(U) = \bigcup_{\mathcal{B}\ni B\subset U} f(B),$$

which is open as the union of open sets. Hence f is an open map. □

(c) We first note that this function f is not open because $f((-\frac{1}{2}, \frac{1}{2})) = (0, \frac{1}{2}) \cup \{1\}$, which is not open in $\mathbb{R}$. However, the open rays in $\mathbb{R}$ are a subbasis for the order topoology on $\mathbb{R}$, which is the same as the standard topology. Observe that

$$f((a, +\infty)) = \begin{cases} (a, +\infty) & \text{if } a > 0 \\ (0, +\infty) & \text{otherwise} \end{cases}$$

and

$$f((-\infty, a)) = \begin{cases} (|a|, +\infty) & \text{if } a < 0 \\ (0, +\infty) & \text{otherwise} \end{cases}$$

Thus f maps the subbasis of open rays to open sets, but it is not an open map. □

5. (a) Denote this set by A. We will $\{\frac{1}{n}\}$ is open in A for each $n \in \mathbb{N}$. Note that $\{1\} = A \cap (3/4, +\infty)$ and so is open in A. For $n \in \mathbb{N}$ with $n > 1$, let $\epsilon := \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n}$. Then $\frac{1}{n+1} < \frac{1}{n} - \frac{\epsilon}{2}$ and $\frac{1}{n-1} > \frac{1}{n} + \frac{\epsilon}{2}$. Consequently,

$$\left\{\frac{1}{n}\right\} = A \cap \left(\frac{1}{n} - \frac{\epsilon}{2}, \frac{1}{n} + \frac{\epsilon}{2}\right)$$

and so is open in A. □

(b) Denote this set by B. We claim that $\{0\}$ is not open in B. If it was open, then there would exist an open subset U of $\mathbb{R}$ satisfying $\{0\} = B \cap U$. However, $0 \in U$ implies there exists $\epsilon > 0$ satisfying $(0 - \epsilon, 0 + \epsilon) = (-\epsilon, \epsilon) \subset U$. Let $n \in \mathbb{N}$ be such that $n > \frac{1}{\epsilon}$. Then $0 < \frac{1}{n} < \epsilon$ and hence

$$\frac{1}{n} \in B \cap (-\epsilon, \epsilon) \subset B \cap U,$$

a contradiction. □

6. (a) We first argue that there is at least one such $n \in \mathbb{Z}$. First, suppose there exists $a, b \in \mathbb{Z}$ with $a \leq x \leq b$. Let n be the largest integer in $\{a, a+1, a+2, \ldots, a+(b-a) = b\}$ satisfying $n \leq x$. Consequently, $n + 1 > x$ and hence $n \leq x < n+1$. Now, if no such $a, b \in \mathbb{Z}$ exist, then we must have that either x is a lower bound for $\mathbb{Z}$ or an upper bound for $\mathbb{Z}$. We will argue that the latter yields a contradiction (the proof of the former is similar). Indeed, if $\mathbb{Z}$ is bounded above, then it has a least upper bound $y \in \mathbb{R}$. But $y - 1$ cannot be an upper bound for $\mathbb{Z}$ (lest we contradict y being the *least* upper bound) and hence $y - 1 \leq n$ for some $n \in \mathbb{Z}$. Adding 1 to each side of this

inequality yields $y \leq n+1$ and so $y < n+2 \in \mathbb{Z}$, which contradicts y being an upper bound for $\mathbb{Z}$. Thus we must always be in the first case, where we found $n \leq x < n+1$.

Now, suppose $n \leq x < n+1$ and $m \leq x < m+1$ for $n, m \in \mathbb{Z}$. We must show $n = m$. If not, then (without loss of generality) $n < m$. Consequently $n+1 \leq m$, which implies $x < n+1 \leq m \leq x$, which contradicts the strict inequality $x < n+1$. □

(b) Note that $x - y > 1$ implies $y < y+1 < x$. Let $n \in \mathbb{Z}$ be the unique integer satisfying $n \leq x < n+1$. We consider two cases: $n = x$ and $n < x$. In the former case, we claim $y < n-1 < x$. The second inequality is immediate and if the first inequality fails then $n - 1 \leq y$ which is equivalent to $x - y \leq 1$ (using $x = n$), which contradicts $x - y > 1$. In the case when $n < x$, we claim $y < n < x$. Again the second inequality is immediate and if the first fails then we have $n \leq y$ which implies $n+1 \leq y+1 < x$, contradicting $x < n+1$. □

(c) Let $n \in \mathbb{N}$ be such that $n > \frac{1}{x-y}$ (which exists since otherwise $\mathbb{Z}$ is bounded above we obtain the same contradiction as in part (a)). It follows that $nx - ny = n(x-y) > 1$. By part (b), there exists $m \in \mathbb{Z}$ with $ny < m < nx$. Dividing by n yields $y < \frac{m}{n} < x$, and so we take $z := \frac{m}{n} \in \mathbb{Q}$.□

(d) We first show $\mathcal{B}$ is countable. Define a function $f : \mathcal{B} \to \mathbb{Q} \to \mathbb{Q}$ by sending the interval (a, b) to the ordered pair (a, b) (pardon the unfortunate notation). This is clearly an injection. We have also seen that $\mathbb{Q}$ is countable and that finite products of countable sets are countable, hence $\mathbb{Q} \times \mathbb{Q}$ is countable. So by Theorem 7.1 there is an injection $g : \mathbb{Q} \times \mathbb{Q} \to \mathbb{N}$. But then $g \circ f : \mathcal{B} \to \mathbb{N}$ is an injection and hence $\mathcal{B}$ is countable by Theorem 7.1 again.

Next we show $\mathcal{B}$ is a basis. For $x \in \mathbb{R}$, using part (a) there exists $n \in \mathbb{Z}$ so that $n \leq x < n+1$. Consequently, $x \in (n-1, n+1) \in \mathcal{B}$. So every element of $\mathbb{R}$ is contained in a basis set. Next, suppose $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$. Then $B_1 = (a, b)$ and $B_2 = (c, d)$ for some $a, b, c, d \in \mathbb{Q}$. Since B_1 and B_2 are not disjoint (there intersection contains x), we cannot have $b \leq c$ or $d \leq a$. That is $c < b$ and $a < d$. Consequently, we are in one of the four following cases:

$$\begin{cases} a \leq c < b \leq d & \Rightarrow B_1 \cap B_2 = (c, b) \\ a \leq c < d \leq b & \Rightarrow B_1 \cap B_2 = (c, d) \\ c \leq a < d \leq b & \Rightarrow B_1 \cap B_2 = (a, d) \\ c \leq a < b \leq d & \Rightarrow B_1 \cap B_2 = (a, b) \end{cases}$$

In all four cases, $B_3 := B_1 \cap B_2 \in \mathcal{B}$ and so we have $x \in B_3 \subset B_1 \cap B_2$. Thus $\mathcal{B}$ is a basis. □

(e) Let $\mathcal{T}$ denote the standard topology on $\mathbb{R}$ and let $\mathcal{T}'$ denote the topology generated by $\mathcal{B}$. First note that $\mathcal{B} \subset \mathcal{T}$, since the basis consists of open intervals. Hence $\mathcal{T}' \subset \mathcal{T}$. Conversely, let $x, y \in \mathbb{R}$ with $x < y$. We will show $(x, y) \in \mathcal{T}'$ and since open intervals form a basis for the standard topology on $\mathbb{R}$ it will follow that $\mathcal{T} \subset \mathcal{T}'$ and hence $\mathcal{T} = \mathcal{T}'$. For $z \in (x, y)$ we have $z - x > 0$ and $y - z > 0$. Thus by part (c) there exist $a_z, b_z \in \mathbb{Q}$ satisfying $x < a_z < z < b_z < y$. Consequently, we have $z \in (a_z, b_z) \subset (x, y)$ and $(a_z, b_z) \in \mathcal{T}'$. Therefore

$$(x, y) = \bigcup_{z \in (x,y)} (a_z, b_z) \in \mathcal{T}'$$

as claimed. □