Exercises:

§17, 18

1. Prove each of the following topological spaces is Hausdorff.

   (a) A set X with an order relation $<$ and the order topology.

   (b) A product $X \times Y$ with the product topology where X and Y are Hausdorff spaces.

   (c) A subspace $Y \subset X$ with the subspace topology where X is a Hausdorff space.

2. Let X be a topological space. Show that X is Hausdorff if and only if the **diagonal**

$$\Delta := \{(x, x) \mid x \in X\}$$

   is a closed subset of $X \times X$ with the product topology.

3. Consider the collection $\mathcal{T} = \{U \subset \mathbb{R} \mid \mathbb{R} \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

   (a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$. We call this the **finite complement topology**.

   (b) Show that the finite complement topology is T_1: given distinct points $x, y \in \mathbb{R}$ there exists open sets U and V with $x \in U \not\ni y$ and $x \notin V \ni y$.

   (c) Show that the finite complement topology is not Hausdorff.

   (d) Find all the points that the net $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to in the finite complement topology.

4. Let X be a set with two topologies $\mathcal{T}$ and $\mathcal{T}'$ and let $i\colon X \to X$ be the identity function: $i(x) = x$ for all $x \in X$. Equip the domain copy of X with the topology $\mathcal{T}$ and the range copy of X with the topology $\mathcal{T}'$.

   (a) Show that i is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}'$.

   (b) Show that i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

5. Consider the functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x + y \qquad \text{and} \qquad g(x, y) = x - y.$$

   (a) Show that if $\mathbb{R}$ and $\mathbb{R}^2$ are given the standard topologies, then f and g are continuous.

   (b) Suppose $\mathbb{R}$ is given the lower limit topology and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is given the corresponding product topology. Determine and prove the continuity or discontinuity of f and g.

6*. In this exercise you will establish a homeomorphism between the following two subspaces of $\mathbb{R}^2$:

$$X := \mathbb{R}^2 \setminus \{(0, 0)\} \qquad \text{and} \qquad Y := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1\}.$$

   Throughout, $\mathbb{R}^2$ will have the standard topology and X and Y will have their subspace topologies.

   (a) Define a function $\|\cdot\|\colon \mathbb{R}^2 \to [0, +\infty)$ by $\|(x, y)\| = (x^2 + y^2)^{1/2}$. Show that this function is continuous when $[0, +\infty) \subset \mathbb{R}$ is given the subspace topology. [**Hint:** think geometrically.]

   (b) Show that $X = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| > 0\}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| > 1\}$.

   (c) Show that $f\colon X \to \mathbb{R}^2$ defined by $f(x, y) = \frac{1}{\|(x,y)\|}(x, y)$ is continuous.

   (d) Find continuous functions $g\colon X \to Y$ and $h\colon Y \to X$ satisfying $g \circ h(x, y) = (x, y)$ and $h \circ g(x, y) = (x, y)$, and deduce that X and Y are homeomorphic.

Solutions:

1. (a) Let $x, y \in X$ be distinct. If the open interval (x, y) is empty, then the open rays $U := (-\infty, y)$ and $V := (x, \infty)$ are neighborhoods of x and y respectively. Also $U \cap V = (x, y) = \emptyset$, so these neighborhoods are disjoint. If (x, y) is not empty, then let $z \in (x, y)$ and consider the open rays $U := (-\infty, z)$ and $V = (z, \infty)$, which are again neighborhoods of x and y respectively. Also $U \cap V$ consists of those points w satisfying $w < z$ and $z < w$. But this cannot occur in an order relation and so $U \cap V$ must be empty; that is, U and V are disjoint. Thus X is Hausdorff. □

   (b) Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be distinct points. Since the pairs are distinct, we must have either $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, assume $x_1 \neq x_2$. Since X is Hausdorff, there are disjoint neighborhoods U_1 and U_2 of x_1 and x_2, respectively. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2), respectively. Hence $X \times Y$ is Hausdorff. □

   (c) Let $y_1, y_2 \in Y$ be distinct points. Since y_1 and y_2 also belong to X, which is Hausdorff, there exists disjoint open subsets $U_1, U_2 \subset X$ with $y_j \in U_j$, $j = 1, 2$. Consequently, $V_j := Y \cap U_j$ is an open in the subspace topology and contains y_j, $j = 1, 2$. Moreover, $V_1 \cap V_2 = Y \cap U_1 \cap U_2 = \emptyset$. Hence V_1 and V_2 are disjoint neighborhoods of y_1 and y_2, and so Y is Hausdorff. □

2. $(\Rightarrow)$: Assume X is Hausdorff. To show Δ is closed, we will show its complement is open. Observe that

$$O := (X \times X) \setminus \Delta = \{(x, y) \in X \times X \mid x \neq y\}.$$

Thus X being Hausdorff implies that for each $(x, y) \in O$ there are disjoint neighborhoods U_x and V_y of x and y, respectively. Since these sets are disjoint, it follows that $U_x \times V_y$ is disjoint from $X \times X$. Consequently, $U_x \times V_y \subset O$ and so

$$O = \subset_{(x,y) \in O} U_x \times V_y,$$

which shows that O is open and therefore Δ is closed.

$(\Leftarrow)$: Assume that Δ is closed. Let $x, y \in X$ be distinct. This implies $(x, y) \notin \Delta$. Since the complement of Δ is open, there exists a basis set for the product topology $U \times V$ (i.e. $U, V \subset X$ are open) satisfying $(x, y) \in U \times V \subset (X \times X) \setminus \Delta$. Thus $x \in U$ and $y \in V$ so U and V are neighborhoods for x and y respectively. Also, since $U \times V$ belongs to the complement of Δ it follows that they are disjoint. Hence X is Hausdorff. □

3. (a) Since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is finite, we have $\mathbb{R} \in \mathcal{T}$. We also have $\emptyset \in \mathcal{T}$ by assumption. Let $\mathcal{S} \subset \mathcal{T}$ be a subcollection. Observe that

$$\mathbb{R} \setminus \bigcup_{U \in \mathcal{S}} U = \bigcap_{U \in \mathcal{S}} \mathbb{R} \setminus U.$$

   Thus the above is an intersection of finite sets and is therefore finite. This implies $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$. Finally, let $U_1, \ldots, U_n \in \mathcal{T}$. Then

$$\mathbb{R} \setminus (U_1 \cap \cdots \cap U_n) = (\mathbb{R} \setminus U_1) \cup \cdots \cup (\mathbb{R} \setminus U_n).$$

   So the above is a finite union of finite sets and is therefore finite. This implies $U_1 \cap \cdots \cap U_n \in \mathcal{T}$. Thus $\mathcal{T}$ is a topology. □

   (b) Let $x, y \in \mathbb{R}$ be distinct. Note that $U := \mathbb{R} \setminus \{y\}$ and $V := \mathbb{R} \setminus \{x\}$ are both open in the finite complement topology because their complements are singleton sets. Moreover, $x \in U \not\ni y$ and $x \notin V \ni y$. Thus $\mathcal{T}$ is T_1. □

   (c) Consider two non-empty subsets $U, V \subset \mathbb{R}$ which are open in the finite complement topology. Thus $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are finite and we claim that they cannot be disjoint. Indeed, $U \cap V = \emptyset$ implies $V \subset \mathbb{R} \setminus U$ and hence is finite. But since $\mathbb{R} \setminus V$ is finite, this would imply $\mathbb{R} = V \cup (\mathbb{R} \setminus V)$ is finite, a contradiction. Thus any two non-empty open sets cannot be disjoint. In particular, given any distinct points $x, y \in \mathbb{R}$ any neighborhoods U and V of x and y respectively are necessarily non-empty (they contain either x or y) and hence cannot be disjoint. Thus X is not Hausdorff. □

(d) We claim that every point in $\mathbb{R}$ is a limit point of this net. Fix $x \in \mathbb{R}$. Let U be a neighborhood of x. Since U is open and non-empty, we have $\mathbb{R} \setminus U$ is finite. Consequently it can only contain $\frac{1}{n}$ for finitely many $n \in \mathbb{N}$. Let

$$n_0 := 1 + \max\{n \in \mathbb{N} \mid \frac{1}{n} \in \mathbb{R} \setminus U\}.$$

Then for any $n \geq n_0$, we have $\frac{1}{n} \notin \mathbb{R} \setminus U$ and therefore $\frac{1}{n} \in U$. Thus $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to x. Since $x \in \mathbb{R}$ was arbitrary, every point in $\mathbb{R}$ is a limit of this net. □

4. (a) Observe that for any set $A \subset X$, $i^{-1}(A) = A$. So we have

$$\begin{aligned}\mathcal{T} \text{ is finer than } \mathcal{T}' &\Leftrightarrow \mathcal{T}' \subset \mathcal{T}\\ &\Leftrightarrow U \in \mathcal{T} \text{ for all } U \in \mathcal{T}'\\ &\Leftrightarrow i^{-1}(U) \in \mathcal{T} \text{ for all } U \in \mathcal{T}'\\ &\Leftrightarrow i \text{ is continuous.}\end{aligned}$$

□

(b) Since i is always a bijection, it is a homeomorphism if and only if i and i^{-1} are continuous, which by the previous part is equivalent to $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T}' \subset \mathcal{T}$. Thus i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$. □

5. (a) Recall that a map is continuous if and only if it is continuous at all of the points in its domain. Thus it suffices to show f and g are continuous at every $(x, y) \in \mathbb{R}^2$, and we will do so using the characterization of continuity at a point in terms of convergent nets. We will also only prove the continuity of f since the proof for g is similar. Suppose $((x_i, y_i))_{i\in I} \subset \mathbb{R}^2$ is a net converging to some (x, y). We must show the net $(f(x_i, y_i))_{i\in I} = (x_i + y_i)_{i\in I}$ converges to $f(x, y) = x + y$. Let U be a neighborhood of $x + y$. Then there exists an $\epsilon > 0$ so that $(x + y - \epsilon, x + y + \epsilon) \subset U$. Now,

$$V := (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \times (y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2})$$

is a neighborhood of (x, y) and so by the convergence of the net there is some $i_0 \in I$ so that for all $i \geq i_0$ we have $(x_i, y_i) \in V$. We claim that for $i \geq i_0$ we have $f(x_i, y_i) \in U$. Indeed, we have

$$f(x_i, y_i) = x_i + y_i < x + \frac{\epsilon}{2} + y + \frac{\epsilon}{2} = x + y + \epsilon,$$

and

$$f(x_i, y_i) = x_i + y_i > x - \frac{\epsilon}{2} + y - \frac{\epsilon}{2} = x + y - \epsilon.$$

Thus $f(x_i, y_i) \in (x+y-\epsilon, x+y+\epsilon) \subset U$. We have shown that for all $i \geq i_0$ one has $f(x_i, y_i) \in U$. Thus the net $(f(x_i, y_i))_{i\in I}$ converges to $x+y$, and hence f is continuous at (x, y). Since $(x, y) \in \mathbb{R}^2$ was arbitrary, we obtain that f is continuous. □

(b) We claim that f is still continuous, but g is not. We prove the continuity of f using the same strategy as in the previous part. Suppose $((x_i, y_i)_{i\in I} \subset \mathbb{R}^2$ is a net converging to $(x, y) \in \mathbb{R}^2$. We must show the net $(f(x_i, y_i))_{i\in I} = (x_i + y_i)_{i\in I}$ converges to $f(x, y) = x + y$. Let U be a neighborhood of $x + y$. Recall that the lower limit topology has a basis of half-open intervals of the form $[a, b)$. Thus there exists such a half-open interval satisfying $x + y \in [a, b) \subset U$. Let $\epsilon = b - (x + y)$, then we have $x + y \in [x + y, x + y + \epsilon) \subset [a, b) \subset U$. We will show that there exists $i_0 \in I$ so that for all $i \geq i_0$ one has $x_i + y_i \in [x + y, x + y + \epsilon)$. Observe that

$$V := [x, x + \frac{\epsilon}{2}) \times [y, y + \frac{\epsilon}{2})$$

is an open neighborhood of (x, y). Thus there exists $i_0 \in I$ so that $(x_i, y_i) \in V$ for all $i \geq i_0$. Consequently,

$$f(x_i, y_i) = x_i + y_i < x + \frac{\epsilon}{2} + y + \frac{\epsilon}{2} = x + y\epsilon$$

and

$$f(x_i, y_i) = x_i + y_i \geq x + y.$$

Hence $f(x_i, y_i) \in [x + y, x + y + \epsilon) \subset U$ for all $i \geq i_0$. Thus $(f(x_i, y_i))_{i\in I}$ converges to $f(x, y)$, and so f is continuous at (x, y). Since $(x, y) \in \mathbb{R}^2$ was arbitrary, we see that f is continuous.

To see that g is not continuous, consider the net (sequence) $((0, \frac{1}{n}))_{n\in\mathbb{N}}$. We claim that this converges to $(0, 0) \in \mathbb{R}^2$, but its image under g does not converge to $f(0, 0) = 0$. Let U be a neighborhood of $(0, 0)$. The collection of subsets of the form $[a, b) \times [c, d)$ is a basis for the product topology on $\mathbb{R}^2$. Hence there exists such a basis set satisfying $(0, 0) \in [a, b) \times [c, d) \subset U$. In particular, this implies $c \leq 0 < d$ so that we can find $n_0 \in \mathbb{N}$ with $c \leq 0 < \frac{1}{n_0} < d$. Therefore $(0, \frac{1}{n}) \in [a, b) \times [c, d) \subset U$ for all $n \geq n_0$. Thus $((0, \frac{1}{n}))_{n\in\mathbb{N}}$ converges to $(0, 0)$. Now, the interval $[0, 1)$ is an open neighborhood of $f(0, 0) = 0$. However, $g(0, \frac{1}{n}) = 0 - \frac{1}{n}. = -\frac{1}{n}$ fails to be in this neighborhood for any $n \in \mathbb{N}$. Consequently, $(g(0, \frac{1}{n}))_{n\in\mathbb{N}}$ cannot converge to $g(0, 0)$ and so g is not continuous at $(0, 0)$. In particular, g is not continuous. □

6. (a) Since $[0, +\infty)$ is convex, the subspace topology on $[0, +\infty)$ is the same as its order topology. Consequently, the open rays $(-\infty, a)$ and $(a, +\infty)$ for $a \geq 0$ form a subbasis for this topology. Hence it suffices to show their preimages under $\|\cdot\|$ are open. Note that $\|(x, y)\|$ gives the distance in $\mathbb{R}^2$ to the origin, and thus the preimage of $(-\infty, a)$ is the interior of the circle with radius a which we have seen is in open in the standard topology (recall that we actually showed the collection of interiors of circles generated the standard topology on $\mathbb{R}^2$). Also, the preimage of $(a, +\infty)$ is the exterior of the circle of radius a. To see that this is open, let (x, y) be a point in the set and define

$$\epsilon := \|(x, y)\| - a > 0.$$

Note that the circle centered at (x, y) with radius ϵ is tangent to the circle of radius a. Consequently, the interior of the circle with center (x, y) and radius ϵ is an open set that contains (x, y) and is contained in the exterior of the circle of radius a. Since (x, y) was arbitrary, this shows the exterior of the circle of radius a is open, and thus $\|\cdot\|$ is continuous. □

(b) Note that $\|(x, y)\| = 0$ if and only if $x^2 + y^2 = 0^2 = 0$. Since $x^2, y^2 \geq 0$, this sum can give zero if and only if $x = y = 0$. Thus $\|(x, y)\| = 0$ if and only if $(x, y) = (0, 0)$, which means $\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| > 0\} = \mathbb{R}^2 \setminus \{(0, 0)\} = X$. The equality for Y follows from $\|(x, y)\| > 1$ if and only if $x^2 + y^2 = \|(x, y)\|^2 > 1^2 = 1$. □

(c) We first require a lemma (the triangle inequality for $\|\cdot\|$): for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$,

$$\|\mathbf{x} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|,$$

where subtraction is defined entrywise (i.e. vector subtraction). Note that $\|\mathbf{x}-\mathbf{z}\|$ is precisely the usual distance from $\mathbf{x}$ to $\mathbf{z}$. The above inequality will follow from $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\|+\|\mathbf{v}\|$ by choosing $\mathbf{u} = \mathbf{x} - \mathbf{y}$ and $\mathbf{v} = \mathbf{y} - \mathbf{z}$, so we prove this new inequality instead. Squaring and expanding the left-hand side yields

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2.$$

The Cauchy–Schwarz inequality implies the quantity $\mathbf{u} \cdot \mathbf{v}$ is bounded above by $\|\mathbf{u}\|\|\mathbf{v}\|$ (this can also be checked directly by squaring both quantities), and so we have

$$\|\mathbf{u} + \mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.$$

Taking the square root yields $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$, and so we also obtain the original desired inequality.

Now, we will show that f is continuous at every point in X. Fix some $\mathbf{x}_0 \in X$ and let V be a neighborhood of $f(\mathbf{x}_0)$. We must find a neighborhood U of $\mathbf{x}_0$ so that $f(U) \subset V$. First note that there exists an $\epsilon > 0$ so that the interior of the circle with center $f(\mathbf{x}_0)$ and radius ϵ is contained in V. The interior of this circle is precisely the set

$$B := \{\mathbf{y} \in \mathbb{R}^2 \mid \|\mathbf{y} - f(\mathbf{x}_0)\| < \epsilon\}.$$

We will show that there exists some $\delta > 0$ so that $\|\mathbf{x}-\mathbf{x}_0\| < \delta$ implies $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \epsilon$. Since the set of $\mathbf{x}$ satisfying $\|\mathbf{x}-\mathbf{x}_0\| < \delta$ is exactly the interior of the circle with center $\mathbf{x}_0$ and radius δ, it is a neighborhood U of $\mathbf{x}_0$. Thus finding such a δ yields a neighborhood U with $f(U) \subset B \subset V$, and therefore shows f is continuous at $\mathbf{x}_0$. Since $\mathbf{x}_0 \in X$ was arbitrary, this will complete the proof. In order to determine a suitable δ we first require some computations. Using our lemma we have for $\mathbf{x} \in X$

$$\|f(\mathbf{x}) - f(\mathbf{x_0})\| = \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{x}}{\|\mathbf{x}_0\|} + \frac{\mathbf{x}}{\|\mathbf{x}_0\|} - \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} \right\| \leq \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{x}}{\|\mathbf{x}_0\|} \right\| + \left\| \frac{\mathbf{x}}{\|\mathbf{x}_0\|} - \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} \right\|$$

Now, observing that $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for $\alpha \in \mathbb{R}$, we continue our estimate of the above with

$$\begin{aligned} \|f(\mathbf{x}) - f(\mathbf{x_0})\| &\leq \left| \frac{1}{\|\mathbf{x}\|} - \frac{1}{\|\mathbf{x}_0\|} \right| \|\mathbf{x}\| + \frac{1}{\|\mathbf{x}_0\|}\|\mathbf{x}-\mathbf{x}_0\| \\ &= \left| \frac{\|\mathbf{x}_0\| - \|\mathbf{x}\|}{\|\mathbf{x}\|\|\mathbf{x}_0\|} \right| \|\mathbf{x}\| + \frac{1}{\|\mathbf{x}_0\|}\|\mathbf{x}-\mathbf{x}_0\| \\ &= \frac{1}{\|\mathbf{x}_0\|}\left(|\|\mathbf{x}_0\| - \|\mathbf{x}\|| + \|\mathbf{x}-\mathbf{x}_0\| \right). \end{aligned}$$

Now, we claim that $|\|\mathbf{x}_0\| - \|\mathbf{x}\|| \leq \|\mathbf{x}-\mathbf{x}_0\|$ (this is called the reverse triangle inequality. Indeed, from our lemma we have $\|\mathbf{u}+\mathbf{v}\| - \|\mathbf{v}\| \leq \|\mathbf{u}\|$, and choosing $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$ and $\mathbf{v} = \mathbf{x}_0$ yields

$$\|\mathbf{x}\| - \|\mathbf{x}_0\| \leq \|\mathbf{x}-\mathbf{x}_0\|,$$

while choosing $\mathbf{u} = \mathbf{x}_0 - \mathbf{x}$ and $\mathbf{v} = \mathbf{x}$

$$\|\mathbf{x}_0\| - \|\mathbf{x}\| \leq \|\mathbf{x}_0 - \mathbf{x}\|$$

which is equivalent to

$$-\|\mathbf{x}-\mathbf{x}_0\| = -\|\mathbf{x}_0-\mathbf{x}\| \leq \|\mathbf{x}\| - \|\mathbf{x}_0\|.$$

Thus $|\|\mathbf{x}\| - \|\mathbf{x}_0\|| = |\|\mathbf{x}_0\| - \|\mathbf{x}\|| \leq \|\mathbf{x}-\mathbf{x}_0\|$. Consequently we may continue our estimate for f with

$$\|f(\mathbf{x}) - f(\mathbf{x_0})\| \leq \frac{2}{\|\mathbf{x}_0\|}\|\mathbf{x}-\mathbf{x}_0\|.$$

Thus if we choose $\delta := \frac{\epsilon\|\mathbf{x}_0\|}{2} > 0$ (recall that $\mathbf{x}_0 \in X$ implies $\|\mathbf{x}_0\| > 0$), then $\|\mathbf{x}-\mathbf{x}_0\| < \delta$ implies $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \frac{2}{\|\mathbf{x}_0\|}\delta = \epsilon$. As discussed above, this completes the proof. □

(d) Consider

$$g(\mathbf{x}) := (1 + \|\mathbf{x}\|)f(\mathbf{x}) \qquad \text{and} \qquad h(\mathbf{x}) := (\|\mathbf{x}\| - 1)f(\mathbf{x}).$$

Observe that for $\mathbf{x} \in X$

$$\|g(\mathbf{x})\| = (1 + \|x\|)\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| = (1 + \|\mathbf{x}\|)\frac{\|\mathbf{x}\|}{\|\mathbf{x}\|} = 1 + \|\mathbf{x}\|.$$

This is strictly bigger than 1 since by part (b) we have $\|\mathbf{x}\| > 0$ for all $\mathbf{x} \in X$. Hence $g(\mathbf{x}) \in Y$ by part (b) again. Similarly, for $\mathbf{x} \in Y$ we have $\|h(\mathbf{x})\| = \|\mathbf{x}\| - 1 > 0$, so that $h(\mathbf{x}) \in X$. Thus $g\colon X \to Y$ and $h\colon Y \to X$.

We next check g and h are inverses of one another. Using our above computations we have for $\mathbf{x} \in Y$

$$g \circ h(\mathbf{x}) = (1 + \|h(\mathbf{x})\|)\frac{h(\mathbf{x})}{\|h(\mathbf{x})\|} = (1 + \|\mathbf{x}\| - 1)\frac{(\|\mathbf{x}\| - 1)f(\mathbf{x})}{\|\mathbf{x}\| - 1} = \|\mathbf{x}\|f(\mathbf{x}) = \mathbf{x},$$

and for $\mathbf{x} \in X$ we have

$$h \circ g(\mathbf{x}) = (\|g(\mathbf{x})\| - 1)\frac{g(\mathbf{x})}{\|g(\mathbf{x})\|} = (1 + \|\mathbf{x}\| - 1)\frac{(1 + \|\mathbf{x}|)f(\mathbf{x})}{1 + \|\mathbf{x}\|} = \|\mathbf{x}\|f(\mathbf{x}) = \mathbf{x}.$$

Thus g and h are inverses of one another, and in particular are both bijections.

Finally, it remains to check g and h are continuous. We will once again check continuity at each point. Fix $\mathbf{x}_0 \in X$. Using our lemma and estimates from the previous part we have for $\mathbf{x} \in X$ that

$$\begin{aligned}
\|g(\mathbf{x}) - g(\mathbf{x}_0)\| &= \|(1+\|\mathbf{x}\|)f(\mathbf{x}) - (1+\|\mathbf{x}_0\|)f(\mathbf{x}) + (1+\|\mathbf{x}_0\|)f(\mathbf{x}) - (1+\|\mathbf{x}_0\|)f(\mathbf{x}_0)\| \\
&\leq \|(1+\|\mathbf{x}\|)f(\mathbf{x}) - (1+\|\mathbf{x}_0\|)f(\mathbf{x})\| + \|(1+\|\mathbf{x}_0\|)f(\mathbf{x}) - (1+\|\mathbf{x}_0\|)f(\mathbf{x}_0)\| \\
&= |\|\mathbf{x}\| - \|\mathbf{x}_0\|| + (1+\|\mathbf{x}_0\|)\|f(\mathbf{x}) - f(\mathbf{x}_0)\| \\
&\leq \|\mathbf{x}_0 - \mathbf{x}\| + (1+\|\mathbf{x}_0\|)\frac{2}{\|\mathbf{x}_0\|}\|\mathbf{x} - \mathbf{x}_0\| \\
&= \left(3 + \frac{2}{\|\mathbf{x}_0\|}\right)\|\mathbf{x} - \mathbf{x}_0\|.
\end{aligned}$$

So by proceeding in the same manner as in the previous part, given $\epsilon > 0$ we can choose $\delta := \left(3 + \frac{2}{\|\mathbf{x}_0\|}\right)^{-1}\epsilon > 0$ to show that g is continuous at $\mathbf{x}_0$ and hence continuous everywhere. A similar string of inequalities to the above yields for $\mathbf{x}, \mathbf{x}_0 \in Y$ that

$$\|h(\mathbf{x}) - h(\mathbf{x}_0)\| \leq \left(3 - \frac{2}{\|\mathbf{x}_0\|}\right)\|\mathbf{x} - \mathbf{x}_0\|.$$

Note that $\mathbf{x}_0 \in Y$ implies $\|\mathbf{x}_0\| > 1$ and so the first factor on the right side above is greater than $3 - 2 = 1$, and so in particular is positive. Thus, as before given some $\epsilon > 0$ we can choose $\delta := \left(3 - \frac{2}{\|\mathbf{x}_0\|}\right)^{-1}\epsilon > 0$ to show that h is continuous at $\mathbf{x}_0$ and hence continuous everywhere. Thus X and Y are homeomorphic. $\square$