Exercises:

§18, 19

1. Let A, B, C, D be topological spaces and suppose $f\colon A \to B$ and $g\colon C \to D$ are continuous functions. Define a function $f \times g\colon A \times C \to B \times D$ by

$$(f \times g)(a, c) = ((f(a), g(c)).$$

Show that $f \times g$ is continuous when $A \times C$ and $B \times D$ are given the product topologies.

2. Let $\mathbb{R}$ and $\mathbb{R}^2$ have their standard topologies.
   (a) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$ is continuous.
   (b) For each $n \in \mathbb{N}$, show that $p\colon \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^n$ is continuous.

3. Let X be a topological space and let Y be set with order relation $<$ and the order topology. Suppose $f, g\colon X \to Y$ are continuous.
   (a) Show that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X.
   (b) Show that the function $h\colon X \to Y$ defined by $h(x) := \min\{f(x), g(x)\}$ is continuous. [**Hint:** using the pasting lemma.]

4. Let $\mathbb{R}$ have the standard topology. Consider

$$C = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid x_n \neq 0 \text{ for only finitely many } n \in \mathbb{N}\}.$$

That is, C is the set of sequences that are eventually equal to zero.
   (a) Determine $\overline{C}$ when $\mathbb{R}^{\mathbb{N}}$ has the box topology.
   (b) Determine $\overline{C}$ when $\mathbb{R}^{\mathbb{N}}$ has the product topology.

5. Let $\{X_j \mid j \in J\}$ be an indexed family of topological spaces. Let $(\mathbf{x}_i)_{i\in I} \subset \prod_{j\in J} X_j$ be a net; that is, for each i in the directed set I, $\mathbf{x}_i \in \prod_{j\in J} X_j$ is a J-tuple.
   (a) Equip $\prod_{j\in J} X_j$ with the product topology and show that the net $(\mathbf{x}_i)_{i\in I}$ converges to some $\mathbf{x} \in \prod_{j\in J} X_j$ if and only if for every $j \in J$ the net $(\pi_j(\mathbf{x}_i))_{i\in I}$ converges to $\pi_j(\mathbf{x})$ in X_j.
   (b) Equip $\prod_{j\in J} X_j$ with the box topology and prove one of the directions in the previous part is true and show the other is false by finding a counterexample in $\mathbb{R}^{\mathbb{N}}$.

6*. Let $\mathbb{R}$ have the standard topology and consider the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

and

$$g(x) = \begin{cases} \frac{1}{m} & x \in \mathbb{Q} \text{ with } x = \frac{n}{m} \text{ for } n \in \mathbb{Z} \text{ and } m \in \mathbb{N} \text{ sharing no common factors} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

   (a) Show that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.
   (b) Show that f is **not** continuous at any $x \in \mathbb{R}$.
   (c) Show that g is **not** continuous at any $x \in \mathbb{Q}$.
   (d) Show that g is continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$.

Solutions:

1. Recall that $\mathcal{B} := \{U \times V \mid U \subset B,\ V \subset D \text{open}\}$ is a basis for the product topology on $B \times D$. So by a proposition from lecture it suffices to show $(f \times g)^{-1}(U \times V)$ is open for all $U \times V \in \mathcal{B}$. Observe that $(f \times g)(a, c) \in U \times V$ if and only if $f(a) \in U$ and $g(c) \in V$, which is in turn equivalent to $(a, c) \in f^{-1}(U) \times g^{-1}(V)$. Thus

$$(f \times g)^{-1}(U, V) = f^{-1}(U) \times g^{-1}(V).$$

Since f and g are both continuous, we know $f^{-1}(U) \subset A$ and $g^{-1}(V) \subset C$ are open. Thus the above set is open in $A \times C$ under the product topology, and therefore $f \times g$ is continuous. □

2. (a) We will check continuity at each point. Fix $(x_0, y_0) \in \mathbb{R}^2$ and let V be a neighborhood of $f(x_0, y_0) = x_0 y_0$. We must find a neighborhood U of (x_0, y_0) satisfying $f(U) \subset V$. By Exercise 1 on Homework 3 there exists $\epsilon > 0$ so that $(x_0 y_0 - \epsilon, x_0 y_0 + \epsilon)$. Observe that for $(x, y) \in \mathbb{R}^2$ we have

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| &= |xy - x_0y_0| \\ &= |xy - xy_0 + xy_0 - x_0y_0| \\ &\leq |x||y - y_0| + |x - x_0||y_0| \\ &= |x - x_0 + x_0||y - y_0| + |x - x_0||y_0| \\ &\leq (|x - x_0| + |x_0|)|y - y_0| + |x - x_0||y_0|. \end{aligned}$$

Thus if we let $\delta := \min\{\frac{\epsilon}{2(1+|x_0|+|y_0|)}, 1\}$ then for

$$(x, y) \in U := (x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)$$

the above estimates show

$$|f(x,y) - f(x_0,y_0)| \leq (\delta + |x_0|)\delta + \delta|y_0| \leq (1 + |x_0|)\delta + \delta|y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f(x, y) \in V$ and so $f(U) \subset V$ as needed. Since $(x_0, y_0) \in \mathbb{R}^2$ was arbitrary, we see that f is continuous. □

(b) We will proceed by induction on n, but first need another function. Consider $g\colon \mathbb{R} \to \mathbb{R}^2$ defined by $g(x) = (x, x)$. Then for open sets $U, V \subset \mathbb{R}$ one has $g(x) \in U \times V$ if and only if $x \in U \cap V$. Thus $g^{-1}(U \times V) = U \cap V$ which is open. Since such products form a basis for the topology on $\mathbb{R}^2$, we see that g is continuous.

Now, for $n = 2$, we have $p(x) = f \circ g$. Since f is continuous by part (a), we see that p is a composition of continuous functions and is therefore continuous. Suppose $q(x) = x^{n-1}$ is continuous. Then $p = f \circ (q \times i) \circ g$ where $i(x) = x$ is the identity function. Since q and i are continuous, so is $q \times i$ by the Exercise 1. Thus p is continuous as the composition of continuous functions. □

3. (a) Denote the set in question by A. We will show A is closed by showing its complement is open. If $x \in X \setminus A$ then we must have $f(x) > g(x)$. We will find an open set satisfying $x \in U_x \subset X \setminus A$. If the open interval $(g(x), f(x))$ is nonempty, let $y \in (g(x), f(x))$ and consider $U_x := g^{-1}((-\infty, y)) \cap f^{-1}((y, +\infty))$. Then U_x is open by the continuity of f and g and it clearly contains x. Also, if $x' \in U_x$ then we have $g(x') < y$ and $f(x') > y$ so that $g(x') < f(x')$ by transitivity of the order. Consequently, $x' \in X \setminus A$ and so $U_x \subset X \setminus A$. If, on the other hand, the open interval $(g(x), f(x))$ is empty, then let $U_x := g^{-1}((-\infty, f(x)) \cap f^{-1}((g(x), \infty))$. This is again open by the continuity of f and g and it contains x. Also if $x' \in U_x$, then $g(x') < f(x)$ and $f(x') > g(x)$. If we had $f(x') \leq g(x')$ then it would follow that $g(x) < f(x') < f(x)$, contradicting $(g(x), f(x)) = \emptyset$. Thus we must have $f(x') > g(x')$, which means $x' \in X \setminus A$ and $U_x \subset X \setminus A$. Thus in either case, we have found our desired open set and so

$$X \setminus A = \bigcup_{x \in X \setminus A} U_x$$

is open as a union of open sets. □

(b) Let $A := \{x \in X \mid f(x) \leq g(x)\}$ and $B := \{x \in X \mid g(x) \leq f(x)\}$. Then these are closed subsets of X by the previous part (where we simply swap the roles of f and g for B), and moreover there union is X: for all $x \in X$ we must have either $f(x) < g(x)$, $f(x) = g(x)$, or $f(x) > g(x)$ and so A contains all those x satisfying the first two while B contains all those x satisfying the last two. Observe that for $x \in A$ we have $h(x) = f(x)$, and for $x \in B$ we have $h(x) = g(x)$. Thus $h\mid_A = f$ and $h\mid_B = g$, which are continuous by assumption. Also note that for $x \in A \cap B$ we have $f(x) \leq g(x)$ and $g(x) \leq f(x)$ which implies $f(x) = g(x)$. Thus $f\mid_{A\cap B} \equiv \mid_{A\cap B}$ and so the pasting lemma implies h is continuous. □

4. (a) We claim that C is closed and hence equals its own closure. We will show $\mathbb{R}^{\mathbb{N}} \setminus C$ is open. Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \setminus C$. Consider $A := \{n \in \mathbb{N} \mid x_n \neq 0\}$, which is infinite by virtue of $\mathbf{x} \notin C$. For $n \in A$, define $U_n := (x_n - |x_n|, x_n + |x_n|)$ and note that $0 \notin U_n$. For $n \in \mathbb{N} \setminus A$, define $U_n := \mathbb{R}$. Then then product

$$U := \prod_{n\in\mathbb{N}} U_n$$

is a neighborhood of $\mathbf{x}$ in the box topology, but we claim it is disjoint from C. Indeed, for any $\mathbf{y} = (y_n)_{n\in\mathbb{N}} \in C$ we have $y_n \neq 0$ for each $n \in A$. Since A is infinite, we see that $\mathbf{y} \notin C$. We have shown that every element of $\mathbb{R}^{\mathbb{N}} \setminus C$ has a neighborhood entirely contained in this set. Hence $\mathbb{R}^{\mathbb{N}} \setminus C$ is open and therefore C is closed. □

(b) We claim that $\overline{C} = \mathbb{R}^{\mathbb{N}}$. It suffices to show for any $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, all of its neighborhoods intersect C. Let U be a neighborhood of $\mathbf{x}$. Then there exists a sequence of open sets $U_n \subset \mathbb{R}$ with $U_n = \mathbb{R}$ for all but finitely many $n \in \mathbb{N}$ satisfying

$$\mathbf{x} \in \prod_{n\in\mathbb{N}} U_n \subset U$$

(this is because such sets form a basis for the product topology). Consider the finite set $A := \{n \in \mathbb{N} \colon U_n \neq \mathbb{R}\}$, and define $\mathbf{y} = (y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ by

$$y_n := \begin{cases} x_n & \text{if } n \in A \\ 0 & \text{otherwise} \end{cases}.$$

Since A is finite, we have $\mathbf{y} \in C$ and moreover $\mathbf{y} \in \prod U_n \subset U$. Thus $U \cap C \neq \emptyset$ and so $\mathbf{x} \in \overline{C}$. Since $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ was arbitrary, we see that $\overline{C} = \mathbb{R}^{\mathbb{N}}$ as claimed. □

5. (a) ($\Longrightarrow$): Recall that we showed continuous functions map convergent nets to convergent nets. Since the coordinate projections π_j are continuous for each $j \in J$, it follows that $(\pi_j(\mathbf{x}_i))_{i\in I}$ converges to $\pi_j(\mathbf{x})$.

($\Longleftarrow$): Let U be a neighborhood of $\mathbf{x}$. Then there exists an indexed collection of open sets $\{U_j \subset X_j \mid j \in J\}$ such that $U_j = X_j$ for all but finitely many $j \in J$ satisfying

$$\mathbf{x} \in \prod_{j\in J} U_j \subset U.$$

Thus U_j is a neighborhood for $\pi_j(\mathbf{x})$ for each $j \in J$. Let $J_0 := \{j \in J \mid U_j \subsetneq X_j\}$. For each $j \in J_0$, there exists $i_j \in I$ so that for $i \geq i_j$ one has $\pi_j(\mathbf{x}_i) \in U_j$. Using the upper bound property of the directed set I (and induction), there exists $i_0 \in I$ so that $i_j \leq i_0$ for all $j \in J_0$ (this very much uses that J_0 is finite). Thus for $i \geq i_0$, we have $\pi_j(\mathbf{x}_i) \in U_j$ for all $j \in J_0$ since $i \geq i_0 \geq i_j$. For $j \notin J_0$, we also have $\pi_j(\mathbf{x}_i) \in U_j$ for $i \geq i_0$ simply because $U_j = X_j$ in this case. Consequently, for $i \geq i_0$ we have

$$\mathbf{x}_i \in \prod_{j\in J} U_j \subset U.$$

Thus the net $(\mathbf{x}_i)_{i\in I}$ converges to $\mathbf{x}$. □

(b) The "$\Longrightarrow$" direction in the previous part is still true in the box topology: this topology is finer than the product topology which implies the coordinate projections are still continuous in this case and hence the same proof works. The other direction is false as the following counterexample demonstrates.

Consider

$$\mathbf{x} = (1, 1, 1, \ldots) \in \mathbb{R}^{\mathbb{N}}$$

and let $\mathbf{x}_n \in \mathbb{R}^{\mathbb{N}}$ be the sequence whose first n entries are 1 and the rest are zero. For each $m \in \mathbb{N}$, we have

$$\pi_m(\mathbf{x}_n) = \begin{cases} 1 & \text{if } m \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Thus for all $m \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \pi_m(\mathbf{x}_n) = 1 = \pi_m(\mathbf{x}).$$

However, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ belongs to the set C from Exercise 4, while $\mathbf{x} \notin C$. We showed in 4.(a) that C is closed and thus this net cannot converge to a point outside of it. So $(\mathbf{x}_n)_{n\in\mathbb{N}}$ does not converge to $\mathbf{x}$. □

6. (a) Let $x \in \mathbb{R}$ and let U be a neighborhood of x. We must show U intersects $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. Recall that there exists $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subset U$. By Exercise 6*.(c) on Homework 3, there exists $z \in \mathbb{Q}$ with $x - \epsilon < z < x + \epsilon$, and thus $z \in U$. We also know there exists a $w \in \mathbb{Q}$ with $x - \epsilon - \sqrt{2} < w < x + \epsilon - \sqrt{2}$ and therefore $x - \epsilon < w + \sqrt{2} < x + \epsilon$. So $w + \sqrt{2} \in U$ and we claim $w \in \mathbb{R} \setminus \mathbb{Q}$. Indeed, otherwise $w + \sqrt{2} = q \in \mathbb{Q}$ and hence $\sqrt{2} = q - w \in \mathbb{Q}$, contradicting $\sqrt{2}$ being irrational. Thus U also intersects $\mathbb{R} \setminus \mathbb{Q}$. □

(b) Fix $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, then $V := (\frac{1}{2}, \frac{3}{2})$ is a neigbhorhood of $f(x) = 1$. We claim there is no neighborhood U of x satisfying $f(U) \subset V$. Indeed, since $\mathbb{R} \setminus \mathbb{Q}$ is dense we know there always exists $y \in U \cap \mathbb{R} \setminus \mathbb{Q}$, and thus $f(y) = 0 \notin V$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then using $V := (-\frac{1}{2}, \frac{1}{2})$, we can use the density of $\mathbb{Q}$ to find $y \in \mathbb{Q} \cap U$ for any neighborhood U of x and $f(y) = 1 \notin V$. Thus in either case f fails to be continuous at $x \in \mathbb{R}$. □

(c) Fix $x \in \mathbb{Q}$, and suppose $x = \frac{n}{m}$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ sharing no common factors. Thus $V := (0, \frac{2}{m})$ is a neighborhood of $g(x) = \frac{1}{m}$. However, any neighborhood U of x contains some $y \in \mathbb{R} \setminus \mathbb{Q}$ (by the density of the irrationals), and thus $g(y) = 0 \notin V$. Therefore g is not continuous at x. □

(d) Fix $x \in \mathbb{R} \setminus \mathbb{Q}$. We will show g satisfies the $\epsilon - \delta$ definition of continuity. Let $\epsilon > 0$. Let $M \in \mathbb{N}$ satisfy $M \geq \frac{1}{\epsilon}$. Note that for $m \in \{1, \ldots, M\}$, there are only finitely many $n \in \mathbb{Z}$ satisfying $\frac{n}{m} \in [x - 1, x + 1]$ (since this requires $m(x - 1) \leq n \leq m(x + 1)$). Thus the following is a *finite* set:

$$F := \left\{ \frac{n}{m} \in [x - 1, x + 1] \mid 1 \leq m \leq M,\ n \in \mathbb{Z} \right\}.$$

Since $F \subset \mathbb{Q}$, we know $x \notin F$ and therefore

$$\delta := \min_{y\in F} |x - y| > 0.$$

We first claim $\delta < 1$. Indeed, by Exercise 6*.(a) on Homework 3, there exists $n \in \mathbb{Z}$ with $n \leq x < n + 1$. Thus $x - 1 < n$ and $n + 1 \leq x + 1$, which means $n, n + 1 \in F$, and therefore $\delta \leq |x - n| = x - n < 1$.

Now, we claim that if $y \in \mathbb{R}$ satisfies $|x-y| < \delta$, then $|g(x)-g(y)| < \epsilon$. Since $g(x) = 0$, if $y \in \mathbb{R}\setminus\mathbb{Q}$ then $|g(x) - g(y)| = |0 - 0| = 0 < \epsilon$. So now assume $y \in \mathbb{Q}$, and say $y = \frac{n}{m}$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ with no common factors. If $m > M$, then we have

$$|g(x) - g(y)| = |0 - \frac{1}{m}| = \frac{1}{m} < \frac{1}{M} \leq \epsilon,$$

as needed. If $m \leq M$, then since $\delta < 1$ we have $y \in [x - 1, x + 1]$ and hence $y \in F$. But then $|x - y| < \delta \leq |x - y|$ is a contradiction. Thus we cannot have $m \leq M$ and so in all cases we have have show $|g(x) - g(y)| < \epsilon$. □