Exercises:

§21, 22

1. Let X be a set and let Y be a metric space with metric d. Define a metric on Y^X by

$$\overline{\rho}((y_x)_{x\in X}, (z_x)_{x\in X}) := \sup_{x\in X} \overline{d}(y_x, z_x),$$

where $\overline{d}(y,z) = \min\{d(y,z),1\}$ is the standard bounded metric corresponding to d. Let $f_n, f\colon X \to Y$ be functions, $n \in \mathbb{N}$, and define $\mathbf{f_n}, \mathbf{f} \in Y^X$ by $\mathbf{f_n} := (f_n(x))_{x\in X}$ and $\mathbf{f} := (f(x))_{x\in X}$.

   (a) Show that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f if and only if the sequence $(\mathbf{f_n})_{n\in\mathbb{N}}$ converges to $\mathbf{f}$ when Y^X is given the product topology.

   (b) Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f if and only if the sequence $(\mathbf{f_n})_{n\in\mathbb{N}}$ converges to $\mathbf{f}$ when Y^X is given the topology induced by the metric $\overline{\rho}$.

2. Let X be a topological space. For a subset $A \subset X$, a **retraction** of X onto A is a continuous map $r\colon X \to A$ satisfying $r(a) = a$ for all $a \in A$.

   (a) Let $p\colon X \to Y$ be a continuous map between topological spaces. Show that if there exists a continuous function $f\colon Y \to X$ so that $p(f(y)) = y$ for all $y \in Y$, then p is a quotient map.

   (b) Show that a retraction is a quotient map.

3. Consider the following subset of $\mathbb{R}^2$:

$$A := \{(x,y) \in \mathbb{R}^2 \mid \text{either } x \geq 0 \text{ or } y = 0 \text{ (or both)}\}.$$

Define $q\colon A \to \mathbb{R}$ by $q(x,y) = x$. Show that q is a quotient map, but is neither open nor closed.

4. Let X and Y be topological spaces and let $p\colon X \to Y$ be a surjective map.

   (a) Show that a subset $A \subset X$ is saturated with respect to p if and only if $X \setminus A$ is saturated with respect to p.

   (b) Show that $p(U) \subset Y$ is open for all saturated open sets $U \subset X$ if and only if $p(A) \subset Y$ is closed for all saturated closed sets $A \subset X$.

   (c) Show that if p is an injective quotient map, then it is a homeomorphism.

5. Let $X := (0,1] \cup [2,3)$, $Y := (0,2)$, and $Z := (0,1] \cup (2,3)$ and define maps $p\colon X \to Y$ and $q\colon X \to Z$ by

$$p(t) := \begin{cases} t & \text{if } 0 < t \leq 1 \\ t-1 & \text{if } 2 \leq t < 3 \end{cases} \qquad \text{and} \qquad q(t) := \begin{cases} t & \text{if } t \neq 2 \\ 1 & \text{otherwise} \end{cases}.$$

Equip X and Y with their subspace topologies from $\mathbb{R}$ and equip Z with the quotient topology induced by q.

   (a) Show that p is a quotient map.

   (b) Show that q is a quotient map.

   (c) Show that $f\colon Y \to Z$ defined by

$$f(t) := \begin{cases} t & \text{if } 0 < t \leq 1 \\ t+1 & \text{if } 1 < t < 2 \end{cases}$$

   is a homeomorphism. [**Hint:** show $f \circ p = q$.]

6*. Consider

$$X := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$$
$$S^2 := \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\}.$$

In this exercise you will show a quotient space of X is homeomorphic to S^2.

(a) Let $S^1 := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\}$. Show that $f\colon X \setminus S^1 \to \mathbb{R}^2$ defined by

$$f(\mathbf{x}) := \frac{1}{1 - \|\mathbf{x}\|}\mathbf{x}$$

is a homeomorphism.

(b) Show that $g\colon S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ defined by

$$g(\mathbf{x}) := \frac{1}{1 - x_3}(x_1, x_2)$$

is a homeomorphism.

(c) Show that $p\colon X \to S^2$ defined by

$$p(\mathbf{x}) := \begin{cases} g^{-1} \circ f(\mathbf{x}) & \text{if } \mathbf{x} \in X \setminus S^1 \\ (0,0,1) & \text{otherwise} \end{cases}$$

is a quotient map.

(d) Define an equivalence relation on X by $\mathbf{x} \sim \mathbf{y}$ if and only if $p(\mathbf{x}) = p(\mathbf{y})$. Describe the quotient space $X/\sim$ and show that it is homeomorphic to S^2.

Solutions:

1. (a) The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f if and only if $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ for all $x \in X$. Let $\pi_x\colon Y^X \to Y$ be the coordinate projection and note that $\pi_x(\mathbf{f_n}) = f_n(x)$ and $\pi_x(\mathbf{f}) = f(x)$. Thus $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f if and only if $(\pi_x(\mathbf{f_n}))_{n\in\mathbb{N}}$ converges to $\pi_x(\mathbf{f})$ for all $x \in X$. By Exercise 5.(a) on Homework 6, this is further equivalent to $(\mathbf{f_n})_{n\in\mathbb{N}}$ converges to $\mathbf{f}$ in the product topology on Y^X. $\square$

   (b) $(\Longrightarrow)$: Suppose $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. Let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ and all $x \in X$ we have

$$d(f_n(x), f(x)) < \frac{\epsilon}{2}.$$

   The above further implies $\overline{d}(f_n(x), f(x)) < \frac{\epsilon}{2}$. Consequently, for $n \geq n_0$ we have

$$\overline{\rho}(\mathbf{f_n}, \mathbf{f}) = \sup_{x\in X} \overline{d}(f_n(x), f(x)) \leq \frac{\epsilon}{2} < \epsilon.$$

   Thus $(\mathbf{f_n}))_{n\in\mathbb{N}}$ converges to $\mathbf{f}$ in the topology induced by $\overline{\rho}$.

   $(\Longleftarrow)$: Suppose $(\mathbf{f_n}))_{n\in\mathbb{N}}$ converges to $\mathbf{f}$ in the topology induced by $\overline{\rho}$. Let $0 < \epsilon < 1$. Then there exists $n_0 \in \mathbb{N}$ so that for $n \geq n_0$ we have $\overline{\rho}(\mathbf{f_n}, \mathbf{f}) < \epsilon$. Consequently $\overline{d}(f_n(x), f(x)) < \epsilon$ for all $x \in X$. Since $\epsilon < 1$, this implies $d(f_n(x), f(x)) < \epsilon$. That is, for all $n \geq n_0$ and all $x \in X$ we have $d(f_n(x), f(x)) < \epsilon$. Thus $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. $\square$

2. (a) First note that p is surjective: given any $y \in Y$ we have $p(f(y)) = y$ and so $p(X) = Y$. Now, let $V \subset Y$. If V is open, then the continuity of p implies $p^{-1}(V) \subset X$ is open. Conversely, assume $p^{-1}(V) \subset X$ is open. We claim that $f^{-1}(p^{-1}(V)) = V$. Indeed, for $y \in V$ we have $p \circ f(y) = y$ and thus $y \in (p \circ f)^{-1}(V) = f^{-1}(p^{-1}(V))$. Conversely, if $y \in f^{-1}(p^{-1}(V))$ then $y = p(f(y)) \in V$. Thus $V = f^{-1}(p^{-1}(V))$ and so the continuity of f implies V is open since $p^{-1}(V)$ is open. We have shown V is open if and only if $p^{-1}(V)$ is open, and so p is a quotient map. $\square$

   (b) Let $r\colon X \to A$ be a retraction. Recall that the inclusion map $i\colon A \to X$ defined by $i(a) = a$ is continuous. We also have $r(i(a)) = r(a) = a$ for all $a \in A$. Thus r is a quotient map by the previous part. $\square$

3. First observe that $(x, 0) \in A$ for all $x \in \mathbb{R}$, and thus $q(x, 0) = x \in q(A)$ for all $x \in \mathbb{R}$. That is, q is surjective. Since q is the restriction of a coordinate projection, it is continuous. By a proposition from §22, to show q is a quotient map it suffices to show $q(U)$ is open for all saturated open sets $U \subset A$. Let $U \subset A$ be a saturated open set and let $x_0 \in q(U)$. Then there exists $(x_0, y) \in U$. We claim that $(x_0, 0) \in U$. Indeed, if $x_0 < 0$, then we must have $y = 0$ otherwise $(x_0, y) \notin A$. If $x_0 \geq 0$, then since $q(x_0, 0) = x_0 = q(x_0, y)$ we have
$$(x_0, 0) \in q^{-1}(q(U)) = U,$$
since U is saturated. Now, since U is open in A, there exists an open subset $V \subset \mathbb{R}^2$ with $U = V \cap A$. Since the square metric on $\mathbb{R}^2$ induces its standard topology, it follows that there is $\epsilon > 0$ so that
$$(x_0 - \epsilon, x_0 + \epsilon) \times (-\epsilon, \epsilon) \subset V.$$
Thus $(x_0 - \epsilon, x_0 + \epsilon) \times \{0\} \subset V \cap A = U$. Consequently, $(x_0 - \epsilon, x_0 + \epsilon) \subset q(U)$. Since $x_0 \in q(U)$ was arbitrary, it follows that $q(U)$ is open. Thus q is a quotient map.

   Alternatively, one can also simply note that $f : \mathbb{R} \to A$ defined by $f(x) = (x, 0)$ is continuous (since its coordinate functions are continuous) and satisfies $q(f(x)) = q(x, 0) = x$ for all $x \in \mathbb{R}$. Thus q is a quotient map by Exercise 2.(a).

   To see that q is not an open map, consider $U := [0, 1) \times (1, 2) \subset A$. This is open in A since $U = ((-1, 1) \times (1, 2)) \cap A$. However, $q(U) = [0, 1)$ which is not open in $\mathbb{R}$.

   To see that q is not a closed map, consider $B := \{(x, \frac{1}{x}) \in \mathbb{R}^2 \mid x > 0\} \subset A$. To see that this set is closed in $\mathbb{R}^2$ (and hence in A), suppose $(x_i, \frac{1}{x_i})_{i \in I} \subset B$ is a net converging to some $(x_0, y_0) \in \mathbb{R}^2$. Since the coordinate projections are continuous, we see that the nets $(x_i)_{i \in I}$ and $(\frac{1}{x_i}))_{i \in I}$ converge to x_0 and y_0, respectively. By a lemma from §21 we know that $f(t) := \frac{1}{t}$ is continuous for $t > 0$. Consequently, $(\frac{1}{x_i})_{i \in I} = (f(x_i))_{i \in I}$ converges to $f(x_0) = \frac{1}{x_0}$. Since $\mathbb{R}^2$ is Hausdorff and has unique limits, it must be that $y_0 = \frac{1}{x_0}$ and so $(x_0, y_0) \in B$. Thus B is closed. However, $q(B) = (0, \infty)$ which is not closed in $\mathbb{R}$. □

4. (a) The symmetry between A and $X \setminus A$ means it suffices to show the "only if" direction. Suppose A is saturated with respect to p. We first claim that $p(X \setminus A) = Y \setminus p(A)$. If $y \in p(X \setminus A)$, then $y = p(x)$ for some $x \in X \setminus A$. If $y \in p(A)$, then $x \in p^{-1}(p(A)) = A$, a contradiction, and so we must have $y \in Y \setminus p(A)$. Conversely, if $y \in Y \setminus p(A)$ then by surjectivity there exists $x \in X$ with $p(x) = y$. If $x \in A$ then we would have $y \in p(A)$, a contradiction, and so it must be that $x \in X \setminus A$. Thus $y = p(x) \in P(X \setminus A)$, which proves the claim. Using the claim we see that
$$p^{-1}(p(X \setminus A)) = p^{-1}(Y \setminus p(A)) = p^{-1}(Y) \setminus p^{-1}(p(A)) = X \setminus A.$$
Thus $X \setminus A$ is saturated. □

   (b) $(\Longrightarrow)$: Suppose $p(U) \subset Y$ is open for all saturated open sets $U \subset X$. Let $A \subset X$ be a saturated closed set. Then $U := X \setminus A$ is saturated by the previous part and is open. Thus $p(U) = p(X \setminus A)$ is open, but our claim from the previous part implies this equals $Y \setminus p(A)$. Thus $p(A)$ is closed.
   $(\Longleftarrow)$: This follows by changing all instances of "closed" to "open" and vice-versa in the previous argument. □

   (c) Let $p : X \to Y$ be an injective quotient map. Then p is a continuous bijection and so it remains to show its inverse, call it q, is continuous. Observe that since p is injective, $p^{-1}(p(A)) = A$ for all $A \subset X$ by Exercise 1 on Homework 1. That is, all subsets are saturated. Let $U \subset X$ be open, then $q^{-1}(U) = p(U)$. Since U is a saturated open set and p is a quotient map, a proposition from §22 implies $p(U)$ is open. Thus q is continuous and therefore p is a homeomorphism. □

5. (a) We first show p is surjective. Let $y \in Y$. If $y \in (0, 1]$ then $p(y) = y$, and otherwise $p(y + 1) = y$. The fact that p is continuous follows from the pasting lemma: $A := (0, 1]$ and $B := [2, 3)$ are closed in X, their union is all of X, $p\mid_A(t) = t$ and $p\mid_B(t) = t - 1$ are continuous, and since $A \cap B = \emptyset$ there is no overlap to check. Let $U \subset X$ be a saturated open set, and let $y \in p(U)$.

If $y = 1$, then $p(1) = 1 = p(2)$ implies $1, 2 \in p^{-1}(p(U)) = U$. Thus there exists $\epsilon > 0$ so that $(1 - \epsilon, 1] \cup [2, 2 + \epsilon) \subset U$. Thus

$$(1 - \epsilon, 1 + \epsilon) = p((1 - \epsilon, 1] \cup [2, 2 + \epsilon)) \subset p(U).$$

If $y \neq 1$, then $y = p(x)$ for some $x \in (0, 1) \cup (2, 3)$, and $x \in p^{-1}(p(U)) = U$. Thus there exists an $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subset U$. It follows that $(y - \epsilon, y + \epsilon) = p((x - \epsilon, x + \epsilon)) \subset p(U)$. Thus in either case we have shown that there is a neighborhood of y that lies inside of $p(U)$. Since $y \in p(U)$ was arbitrary, this implies $p(U)$ is open. A proposition from §22 then implies p is a quotient map. □

(b) This follows from the fact that Z has the quotient topology induced by q: $V \subset Z$ is open if and only if belongs to the topology which is the collection $\{U \subset Z \mid q^{-1}(U) \subset X \text{ is open}\}$. □

(c) We observe that for $t \in (0, 1]$ we have $p(t) = t \in (0, 1]$, and hence $f(p(t)) = f(t) = t = q(t)$. For $t = 2$, $f(p(2)) = f(1) = 1 = q(2)$. For $t \in (2, 3)$ we have $p(t) = t - 1 \in (1, 2)$, and hence $f(p(t)) = (t-1)+1 = t = q(t)$. Thus $f \circ p = q$. By a theorem from §22, $q = f \circ p$ being a quotient map implies f is a quotient map. We can also see that f is injective since its inverse is given by the function $g := p\,|_Z$. Thus f is an injective quotient map and therefore a homeomorphism by Exercise 4.(c). □

6*. (a) Since $\|\mathbf{x}\|$ is the distance from $\mathbf{x}$ to the origin in the euclidean metric, we know from Exercise 3 on Homework 8 that this function is continuous. Since subtraction is continuous, we further know $\mathbf{x} \mapsto 1 - \|\mathbf{x}\|$ is continuous and in particular is non-zero on $X \setminus S^1$. Thus the coordinate functions of f are continuous as the quotients of continuous functions (the coordinate projections) by non-zero continuous functions. Consider the function

$$h(\mathbf{x}) := \frac{1}{1 + \|\mathbf{x}\|}\mathbf{x}.$$

Since $\|h(\mathbf{x})\| = \frac{\|\mathbf{x}\|}{1+\|\mathbf{x}\|} < 1$, we see that $h \colon \mathbb{R}^2 \to X \setminus S^1$. It is continuous since its coordinate functions are continuous (by the same argument used on f above). Moreover, for $\mathbf{x} \in \mathbb{R}^2$ we have

$$f \circ h(\mathbf{x}) = \frac{1}{1 - \|h(\mathbf{x})\|}h(\mathbf{x}) = \frac{1}{1 - \frac{\|\mathbf{x}\|}{1+\|\mathbf{x}\|}}\frac{1}{1 + \|\mathbf{x}\|}\mathbf{x} = \frac{1}{1 + \|x\| - \|x\|}\mathbf{x} = \mathbf{x},$$

and for $\mathbf{x} \in X \setminus S^1$ we have

$$h \circ f(\mathbf{x}) = \frac{1}{1 + \|f(\mathbf{x})\|}f(\mathbf{x}) = \frac{1}{1 + \frac{\|\mathbf{x}\|}{1-\|\mathbf{x}\|}}\frac{1}{1 - \|\mathbf{x}\|}\mathbf{x} = \frac{1}{1 - \|x\| + \|x\|}\mathbf{x} = \mathbf{x}.$$

Thus $h = f^{-1}$ and so f is a bijection. Since both f and h are continuous, we see that f is a homeomorphism. □

(b) Observe that $1 - x_3 \neq 0$ for $\mathbf{x} = (x_1, x_2, x_3) \in S^2 \setminus \{(0, 0, 1)\}$. Thus g is continuous since each of its coordinate functions are continuous. For $(x_1, x_2) \in \mathbb{R}^2$, consider

$$k(x_1, x_2) := \left(\frac{2}{x_1^2 + x_2^2 + 1}x_1, \frac{2}{x_1^2 + x_2^2 + 1}x_2, \frac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2 + 1}\right).$$

Then k is continuous because its coordinate functions are and we also note that

$$\begin{aligned}\|k(x_1, x_2)\|^2 &= \frac{4x_1^2 + 4x_2^2 + (x_1^2 + x_2^2 - 1)^2}{(x_1^2 + x_2^2 + 1)^2} = \frac{4x_1^2 + 4x_2^2 + (x_1^2 + x_2^2)^2 - 2(x_1^2 + x_2^2) + 1}{(x_1^2 + x_2^2 + 1)^2}\\ &= \frac{(x_1^2 + x_2^2)^2 + 2(x_1^2 + x_2^2) + 1}{(x_1^2 + x_2^2 + 1)^2} = \frac{(x_1^2 + x_2^2 + 1)^2}{(x_1^2 + x_2^2 + 1)^2} = 1\end{aligned}$$

Thus k is valued in S^2, but since the third coordinate is strictly less than one, we see that $k\colon \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\}$. Finally, we observe that for $\mathbf{x} \in \mathbb{R}^2$

$$\begin{aligned} g \circ k(\mathbf{x}) &= \frac{1}{1 - \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1}} \left(\frac{2}{x_1^2+x_2^2+1} x_1, \frac{2}{x_1^2+x_2^2+1} x_2 \right) \\ &= \frac{1}{x_1^2+x_2^2+1-(x_1^2+x_2^2-1)} (2x_1, 2x_2) = \mathbf{x}, \end{aligned}$$

and for $\mathbf{x} \in S^2 \setminus \{(0,0,1)\}$ we have, using $x_1^2+x_2^2 = 1-x_3^2$, that

$$\begin{aligned} k \circ g(\mathbf{x}) &= k\left(\frac{x_1}{1-x_2}, \frac{x_2}{1-x_3} \right) \\ &= \left(\frac{2}{\frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2} + 1} \frac{x_1}{1-x_3}, \frac{2}{\frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2} + 1} \frac{x_2}{1-x_3}, \frac{\frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2} - 1}{\frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2} + 1} \right) \\ &= \left(\frac{2}{\frac{1-x_3^2}{(1-x_3)^2} + 1} \frac{x_1}{1-x_3}, \frac{2}{\frac{1-x_3^2}{(1-x_3)^2} + 1} \frac{x_2}{1-x_3}, \frac{\frac{1-x_3^2}{(1-x_3)^2} - 1}{\frac{1-x_3^2}{(1-x_3)^2} + 1} \right) \\ &= \left(\frac{2(1-x_3)}{1-x_3^2+(1-x_3)^2} x_1, \frac{2(1-x_3)}{1-x_3^2+(1-x_3)^2} x_2, \frac{1-x_3^2-(1-x_3)^2}{1-x_3^2+(1-x_3)^2} \right) = (x_1, x_2, x_3). \end{aligned}$$

Thus $k = g^{-1}$ and g is a homeomorphism. □

(c) We first note that p is surjective since $g^{-1} \circ f$ is surjective onto $S^2 \setminus \{(0,0,1)\}$ as a composition of homeomorphisms. Now, let $U \subset S^2$. If $(0,0,1) \notin U$, then $U \subset S^2 \setminus \{(0,0,1)\}$ and U is open in $S^2 \setminus \{(0,0,1)\}$ iff $p^{-1}(U)$ is open in $X \setminus S^1$ since $g^{-1} \circ f$ is a homeomorphism. Since $S^2 \setminus \{(0,0,1)\}$ is open in S^2 and $X \setminus S^1$ is open in X, this implies U is open in S^2 if and only if $p^{-1}(U)$ is open in X.

Now assume $(0,0,1) \in U$. First suppose U is open in S^2. We must argue that $p^{-1}(U)$ is open in X. Fix $\mathbf{x}_0 \in p^{-1}(U)$. If $\mathbf{x}_0 \notin S^1$, then $p(\mathbf{x}_0) = g^{-1} \circ f(\mathbf{x}_0) \in U \setminus \{(0,0,1)\}$. Since this set is open and $g^{-1} \circ f$ is continuous at $\mathbf{x}_0$, there exists a neighborhood $V \subset X \setminus S^1$ of x_0 satisfying $p(V) = g^{-1} \circ f(V) \subset U \setminus \{(0,0,1)\}$. Hence $V \subset p^{-1}(U)$. If $\mathbf{x}_0 \in S^1$ then $p(\mathbf{x}_0) = (0,0,1)$. Since $U \ni (0,0,1)$ is open, there exists $\epsilon > 0$ so that

$$(-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times (1-\epsilon, 1+\epsilon) \subset U.$$

Consequently, if $Z := \{\mathbf{x} \in S^2 \mid x_3 > 1-\epsilon\}$ then Z is open and $(0,0,1) \in Z \subset U$. Observe that for $\mathbf{x} \in Z \setminus \{(0,0,1)\}$ we have

$$\|g(\mathbf{x})\| = \frac{\sqrt{x_1^2+x_2^2}}{1-x_3} = \frac{\sqrt{1-x_3^2}}{1-x_3} = \left(\frac{1+x_3}{1-x_3} \right)^{1/2} > \frac{1}{\sqrt{1-\epsilon}},$$

and

$$\|f^{-1}(g(\mathbf{x}))\| = \|h(g(\mathbf{x}))\| = \frac{\|g(\mathbf{x})\|}{1+\|g(\mathbf{x})\|} > \frac{1/\sqrt{1-\epsilon}}{1+1/\sqrt{1-\epsilon}} = \frac{1}{\sqrt{1-\epsilon}+1} =: \delta,$$

where we have used the fact that function $\frac{t}{1+t}$ is monotone. Thus $p^{-1}(Z) = \{\mathbf{x} \in X \mid \delta < \|\mathbf{x}\| \leq 1\}$, which is an open subset of X. Since $(0,0,1) \in Z \subset U$, we consequently have $\mathbf{x}_0 \in p^{-1}(Z) \subset p^{-1}(U)$. We have therefore shown that for any $\mathbf{x}_0 \in p^{-1}(U)$ is there is a neighborhood of x_0 contained in $p^{-1}(U)$; that is, $p^{-1}(U)$ is open.

Conversely, suppose $p^{-1}(U)$ is open and let $\mathbf{x}_0 \in U$. If $x_0 \neq (0,0,1)$, then there is a unique $\mathbf{y} \in p^{-1}(U) \setminus S^1$ with $p(\mathbf{y}) = \mathbf{x}_0$. Since $p^{-1}(U) \subset S^1$ is open, there exists a neighborhood V of $\mathbf{y}$ with $V \subset p^{-1}(U) \setminus S^1$. Then $p(V) = g^{-1} \circ f(V)$ is open since $g^{-1} \circ f$ is a homemorphism and $\mathbf{x}_0 \in p(V) \subset U$. Finally, suppose $x_0 = (0,0,1)$. We claim there exists $0 < \delta < 1$ so that $\{\mathbf{x} \in X \mid \delta < \|\mathbf{x}\| \leq 1\} \subset p^{-1}(U)$, in which case the above estimates imply $\mathbf{x}_0 \in \{\mathbf{x} \in S^2 \mid x_3 >$

$1-\epsilon\} \subset U$ for ϵ satisfying $\frac{1}{\sqrt{1-\epsilon}+1} = \delta$. Now, $S^1 \subset p^{-1}(U)$ since $\mathbf{x}_0 = (0,0,1) \in U$. For $\mathbf{y} \in S^1$ and $0 < r < 1$ consider the set

$$B_{\mathbf{y}}(r) := \{(x_1, x_2) \in X \mid \frac{y_2}{y_1} - (1-r) < \frac{x_2}{x_1} < \frac{y_2}{y_1} + (1-r),\ \|(x_1, x_2)\| > 1-r\}.$$

This is a segment of the annulus with inner radius $1-r$ and outer radius 1 which contains $\mathbf{y}$ and is open in X. Since $p^{-1}(U)$ is open, for $\mathbf{y} \in= S^1$ it is easy to see visually that there exists $0 < r(\mathbf{y}) < 1$ so that $B_{\mathbf{y}}(r(\mathbf{y})) \subset p^{-1}(U)$ (just choose $r(\mathbf{y})$ close enough to 1 so that $B_{\mathbf{y}}(r(\mathbf{y}))$ fits inside a ball centered at $\mathbf{y}$ contained in $p^{-1}(U)$). Thus $\{B_{\mathbf{y}}(r(\mathbf{y})) \mid \mathbf{y} \in S^1\}$ is an open cover for S^1, which is a compact set since it is closed and bounded in $\mathbb{R}^2$. Hence there exists a finite subcover and the desired δ is the smallest $r(\mathbf{y})$ that appears in this finite subcover. Thus every $\mathbf{x}_0 \in U$ has a neighborhood V satisfying $V \subset U$, and so U is open.

We have shown $U \subset S^2$ is open iff $p^{-1}(U) \subset X$ is open, and hence p is a quotient map.

(d) For $\mathbf{x} \in X \setminus S^1$, $[\mathbf{x}] = \{\mathbf{x}\}$. For $\mathbf{x} \in S^1$, $[\mathbf{x}] = S^1$. Thus $X/\sim$ looks like a copy of X where the boundary S^1 is a single point. The fact that $X/\sim$ is homeomorphic to S^2 follows from p being a quotient map and a corollary from §22. □