6.1 Basic Theory of L^p Spaces

Throughout $(X, \mathcal{M}, \mu)$ will be a measure space.

Def For an $\mathcal{M}$-measurable function $f: X \to \mathbb{C}$ and $0 < p < \infty$ we define

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p}$$

For $p = \infty$, we define

$$\|f\|_\infty := \sup\{ R \geq 0 : \mu(\{x \in X : |f(x)| > R\}) > 0 \}$$

For $0 < p \leq \infty$, we define the L^p space on $(X, \mathcal{M}, \mu)$ as

$$L^p(X, \mu) := \{ f: X \to \mathbb{C} : f \text{ is } \mathcal{M}\text{-measurable and } \|f\|_p < \infty \}$$

We also denote this set by $L^p(X, \mathcal{M}, \mu)$ or $L^p(\mu)$. If $(X, \mathcal{P}(X))$ is equipped with the counting measure $\#$, we denote $\ell^p(X) := L^p(X, \mathcal{P}(X), \#)$. In particular, $\ell^p := \ell^p(\mathbb{N})$. □

As with $L^1(X, \mu)$, we will identify two functions in $L^p(X, \mu)$ if they agree μ-almost everywhere.
Note that for all $\varepsilon > 0$,

$$\mu(\{x \in X : |f(x)| > \|f\|_\infty - \varepsilon\}) > 0$$

but for $R \geq \|f\|_\infty$

$$\mu(\{x \in X : |f(x)| > R\}) = 0.$$

This is clear for $R > \|f\|_\infty$ by definition, and for $R = \|f\|_\infty$ we note

$$\{x \in X : |f(x)| > \|f\|_\infty\} = \bigcup_{n=1}^\infty \{x \in X : |f(x)| > \|f\|_\infty + \tfrac{1}{n}\}.$$

Consequently, we also have

$$\|f\|_\infty = \inf\{R \geq 0 : \mu(\{x \in X : |f(x)| > R\}) = 0\} \quad (*)$$

and this minimum is attained.

Remark In light of $(*)$, we see that $L^\infty(X, \mu)$ and $\|\cdot\|_\infty$ only depend on μ through its null sets. Hence if $\nu \sim \mu$, then $L^\infty(X, \nu) = L^\infty(X, \mu)$ and $\|\cdot\|_\infty$ gives the same value. □

Ex ① Let $g: \mathbb{R} \to \mathbb{C}$ be a bounded continuous function. For the measure space $(\mathbb{R}, \mathcal{L}, m)$, one has

$$\|g\|_\infty = \sup_{x \in \mathbb{R}} |g(x)| =: \alpha$$

Indeed, for $\varepsilon > 0$ let $x_0 \in \mathbb{R}$ be such that $|g(x_0)| > \alpha - \varepsilon$. By the continuity of g, there exists $\delta > 0$ so that $|g(x)| > \alpha - 2\varepsilon$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Hence

$$m(\{x \in \mathbb{R} : |g(x)| > \alpha - 2\varepsilon\}) \geq m((x_0 - \delta, x_0 + \delta)) = 2\delta > 0$$

Thus $\|g\|_\infty \geq \alpha - 2\varepsilon$, and letting $\varepsilon \to 0$ yields $\|g\|_\infty \geq \alpha$. On the other hand,

$$m(\{x \in \mathbb{R} : |g(x)| > \alpha\}) = m(\emptyset) = 0$$

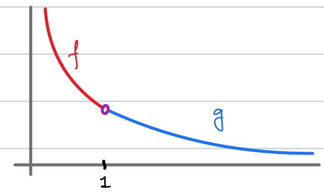
so $\|g\|_\infty \leq \alpha$ by $(*)$.

More generally, let X be a Hausdorff topological space and suppose $(X, \mathcal{B}_X)$ admits a measure μ so that $\mu(U) > 0$ for all non-empty open sets U (μ is called a strictly positive measure). Then the above holds for any continuous $g : X \to \mathbb{C}$.

② For $\alpha > 0$, consider

$$f := \mathbb{1}_{(0,1)} \frac{1}{x^\alpha}$$

$$g := \mathbb{1}_{(1,\infty)} \frac{1}{x^\alpha}$$



For $0 < p < \infty$, we have

$$\|f\|_p = \int_0^1 x^{-\alpha p}\, dx = \begin{cases} \left[\frac{x^{1-\alpha p}}{1-\alpha p}\right]_0^1 & \text{if } \alpha p \neq 1 \\ \left[\ln(x)\right]_0^1 & \text{if } \alpha p = 1 \end{cases} = \begin{cases} \frac{1}{1-\alpha p} & \text{if } 1-\alpha p > 0 \\ \infty & \text{otherwise} \end{cases}$$

so $f \in L^p(\mathbb{R}, m)$ iff $p < \frac{1}{\alpha}$. Similarly

$$\|g\|_p = \int_1^\infty x^{-\alpha p}\, dx = \begin{cases} \left[\frac{x^{1-\alpha p}}{1-\alpha p}\right]_1^\infty & \text{if } \alpha p \neq 1 \\ \left[\ln(x)\right]_1^\infty & \text{if } \alpha p = 1 \end{cases} = \begin{cases} \frac{1}{\alpha p - 1} & \text{if } 1-\alpha p < 0 \\ \infty & \text{otherwise} \end{cases}$$

so $g \in L^p(\mathbb{R}, m)$ iff $p > \frac{1}{\alpha}$.

For fixed p, when α gets too big (e.g. $\alpha \geq \frac{1}{p}$) $f \notin L^p(\mathbb{R}, m)$ because $\frac{1}{x^{\alpha p}}$ grows too quickly as $x \to 0$; whereas when α gets too small (e.g. $\alpha \leq \frac{1}{p}$) $g \notin L^p(\mathbb{R}, m)$ because $\frac{1}{x^{\alpha p}}$ decays too slowly as $x \to \infty$.

For fixed α, the above also shows

$$L^p(\mathbb{R}, m) \not\subset L^q(\mathbb{R}, m) \not\subset L^p(\mathbb{R}, m)$$

for $p < \frac{1}{\alpha} < q$. Heuristically, since $f(x) \to \infty$ as $x \to 0$ this behavior is exacerbated by a larger exponent q, while the $g(x) \to 0$ as $x \to \infty$ is slowed by a smaller exponent p. □

Observe that $L^p(X, \mu)$ for $0 < p \leq \infty$ is a vector space: for $f, g \in L^p(X, \mu)$, $a \in \mathbb{C}$, and $p < \infty$ we have

$$|af + g|^p \leq [2 \max(|a||f|, |g|)]^p \leq 2^p[|a|^p |f|^p + |g|^p]$$

and hence

$$\int_X |af+g|^p \, d\mu \leq 2^p\left[|a|^p \int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu\right] < \infty$$

For $p = \infty$, recall $|f| \leq \|f\|_\infty$ and $|g| \leq \|g\|_\infty$ μ-almost everywhere. So using $|af+g| \leq |a||f| + |g|$ we see that

$$\mu(\{x \in X : |af(x) + g(x)| > |a| \|f\|_\infty + \|g\|_\infty\}) = 0$$

and so $\|af+g\|_\infty \leq |a| \|f\|_\infty + \|g\|_\infty < \infty$ by (*)

Def A norm on a complex vector space V is a map $\|\cdot\| : V \to [0, \infty)$ satisfying:

① $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$. (triangle inequality)

② $\|av\| = |a| \|v\|$ for $a \in \mathbb{C}$, $v \in V$. (homogeneity)

③ $\|v\| = 0$ iff $v = 0$. (positive definite)

We say V is a Banach space if it is complete with respect to the metric $d(v, w) := \|v - w\|$. □

Our main objective is to understand for what values $0<p\le\infty$ is $\|\cdot\|_p$ a norm and $L^p(X,\mu)$ is a Banach space. Homogeneity and positive definiteness are both clear (since we have identified functions up to μ-a.e. equality). So one need only to check the triangle inequality holds, and in that case when $L^p(X,\mu)$ is complete. It turns out both occur iff $p\ge 1$.

To see the triangle inequality fails for $p<1$, observe that for $a,b>0$

$$(a+b)^p - a^p = \int_0^b p(a+t)^{p-1}\,dt < \int_0^b p\,t^{p-1}\,dt = b^p \qquad p-1<0$$

So $(a+b)^p < a^p + b^p$. Now let $E,F\in\mathcal{M}$ be disjoint sets with positive measure. Then for $a:=\mu(E)^{1/p}$ and $b:=\mu(F)^{1/p}$ we have

$$\|1_E + 1_F\|_p = (a^p+b^p)^{1/p} > a+b = \|1_E\|_p + \|1_F\|_p$$

Before proving the triangle inequality holds for $p\ge 1$, we require some intermediate results.

Lemma 6.1 For $a,b\ge 0$ and $0<\lambda<1$, one has

$$a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda)b$$

with equality iff $a=b$.

Proof If $b=0$, then the inequality is $0\le\lambda a + 0$, which is clear and is an equality iff $a=0$. So suppose $b>0$. Dividing both sides of the inequality by b reduces it to

$$\left(\frac{a}{b}\right)^\lambda \le \lambda\frac{a}{b} + (1-\lambda) \qquad **$$

Letting $t:=\frac{a}{b}$, we see that the function $f(t):=t^\lambda - \lambda t$ satisfies $f'(t) = \lambda t^{1-\lambda} - \lambda$. Since $1-\lambda>0$, we have $f'(t)<0$ for $0\le t<1$, $f'(t)>0$ for $t>1$, and $f'(1)=0$. Hence on $[0,\infty)$, f is uniquely maximized at $t=1$; that is,

$$t^\lambda - \lambda t = f(t) \le f(1) = 1-\lambda$$

with equality iff $t=1$ This is equivalent to $(**)$, and $t=1 \iff a=b$. □

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In addition to helping us prove the triangle inequality, the next result is important in its own right.

Theorem 6.2 (Hölder's Inequality) Suppose $1\le p,q\le\infty$ satisfy $\frac{1}{p}+\frac{1}{q}=1$ (where we take the convention that $\frac{1}{\infty}=0$). For $\mathcal{M}$-measurable functions $f,g\colon X\to\mathbb{C}$ one has

$$\|fg\|_1 \le \|f\|_p\|g\|_q.$$

In particular, if $f\in L^p(X,\mu)$ and $g\in L^q(X,\mu)$ then $fg\in L^1(X,\mu)$.

Proof If either $\|f\|_p=0$ or $\|g\|_q=0$ (so that either $f=0$ μ-a.e. or $g=0$ μ-a.e.) then the inequality is an equality. So we now assume $\|f\|_p,\|g\|_q>0$. If either $\|f\|_p=\infty$ or $\|g\|_\infty=\infty$, then again the inequality is clear. So we now assume $0<\|f\|_p,\|g\|_q<\infty$ and denote $\alpha:=\|f\|_p^{\frac{?}{}}$

and $\beta := \|g\|_q^{-1}$.

First suppose $1 < p < \infty$ (and hence $1 < q < \infty$) and let $\lambda := \frac{1}{p} \in (0,1)$. Note $1 - \lambda = \frac{1}{q}$. For $x \in X$, Lemma 6.1 with $a = \alpha^p |f(x)|^p$ and $b = \beta^q |g(x)|^q$ implies

$$\alpha\beta\,|f(x)g(x)| = (\alpha^p |f(x)|^p)^{\frac{1}{p}} (\beta^q |g(x)|^q)^{\frac{1}{q}} \leq \frac{1}{p}\alpha^p |f(x)|^p + \frac{1}{q}\beta^q |g(x)|^q$$

Thus

$$\alpha\beta \int_X |fg|\,d\mu \leq \frac{1}{p}\alpha^p \int_X |f|^p d\mu + \frac{1}{q}\beta^q \int_X |g|^q d\mu$$

$$= \frac{1}{p}\alpha^p \|f\|_p^p + \frac{1}{q}\beta^q \|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1$$

Therefore $\|fg\|_1 \leq (\alpha\beta)^{-1} = \|f\|_p \|g\|_q$.

Now suppose $p = 1$ so that $q = \infty$ (the case $p = \infty$ and $q = 1$ is similar). Then since $|g| \leq \|g\|_\infty$ μ-almost everywhere we have

$$\|fg\|_1 = \int_X |fg|\,d\mu \leq \int_X |f|\,\|g\|_\infty\,d\mu = \|f\|_1 \|g\|_\infty.$$

□

Def We say $1 \leq p, q \leq \infty$ are <u>conjugate exponents</u> if they satisfy $\frac{1}{p} + \frac{1}{q} = 1$. (where $\frac{1}{\infty} = 0$). □

Remark Let $1 \leq p, q \leq \infty$ be conjugate exponents, and let $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$. For $1 < p < \infty$, one has the equality $\|fg\|_1 = \|f\|_p \|g\|_q$ in Hölder's inequality iff

$$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$$

μ-almost everywhere. Indeed, from the proof of Hölder's inequality we see that $\|fg\|_1 = \|f\|_p \|g\|_q$ implies by Proposition 2.16 that

$$\alpha\beta\,|fg| = \frac{1}{p}\alpha^p |f|^p + \frac{1}{q}\beta^q |g|^q$$

μ-almost everywhere, when $\alpha = \|f\|_p^{-1}$ and $\beta = \|g\|_q^{-1}$. Then Lemma 6.1 implies

$$\alpha^p |f|^p = a = b = \beta^q |g|^q$$

μ-almost everywhere. The converse is clear.

For $p = 1$ and $q = \infty$, $\|fg\|_1 = \|f\|_1 \|g\|_\infty$ iff

$$g\,1_{\{x : f(x) \neq 0\}} = \|g\|_\infty$$

This also follows from Proposition 2.16. □

Ex Let $\mu(X) = 1$. For $1 \leq p < q \leq \infty$ we have $L^q(X,\mu) \subset L^p(X,\mu)$.

Indeed set $r := \frac{q}{p} > 1$ and let $1 < s < \infty$ be its conjugate exponent. Then for any $g \in L^q(X,\mu)$ Hölder's inequality implies

$$\int_X |g|^p d\mu = \| |g|^p \|_1 \leq \| |g|^p \|_r \|1\|_s = \| |g|^p \|_r .$$

If $q = \infty$, then $r = \infty$ and so $\| |g|^p \|_\infty = \|g\|_\infty^p < \infty$. For $p < \infty$, $r < \infty$ and so

$$\| |g|^p \|_r = \left(\int_X |g|^{pr} d\mu\right)^{\frac{1}{r}} = \left(\int_X |g|^q d\mu\right)^{p/q} = \|g\|_q^p < \infty$$

So in all cases $g \in L^p(X,\mu)$, and moreover $\|g\|_p \leq \|g\|_q$.

$\boxed{\text{Theorem 6.3}}$ (Minkowski's Inequality) For $1 \leq p \leq \infty$ and $f, g \in L^p(X,\mu)$,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof We already showed this for $p=\infty$ when proving $L^\infty(X,\mu)$ is a vector space, and for $p=1$ this follows from the triangle inequality on $\mathbb{C}$. So we assume $1<p<\infty$. Also, if $f+g=0$ μ-almost everywhere then the inequality is obvious, so we assume this is not the case so that $\|f+g\|_p > 0$.

Now, if $1<q<\infty$ is the conjugate exponent to p, then

$$\frac{1}{p}+\frac{1}{q}=1 \Rightarrow q+p=pq \Rightarrow p=(p-1)q.$$

So applying Hölder's inequality yields

$$\|f+g\|_p^p = \int_X |f+g|^p \, d\mu \leq \int_X (|f|+|g|)|f+g|^{p-1} \, d\mu$$

$$= \int_X |f| \, |f+g|^{p-1} \, d\mu + \int_X |g| \, |f+g|^{p-1} \, d\mu$$

Hölder's inequality

$$\leq (\|f\|_p + \|g\|_p)\left(\int_X |f+g|^{(p-1)q} \, d\mu\right)^{1/q} = (\|f\|_p + \|g\|_p)\, \|f+g\|_p^{p/q}$$

Hence

$$\|f+g\|_p = \|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p.$$

$\boxed{\text{Theorem 6.4}}$ For $1 \leq p \leq \infty$, $L^p(X,\mu)$ is a Banach space.

Proof Let $(f_n)_{n\in\mathbb{N}} \subset L^p(X,\mu)$ be a Cauchy sequence. Then we can find integers $n_1 < n_2 < \cdots$ such that $\|f_n - f_m\|_p < 2^{-j}$ for $n,m \geq n_j$. Denote $g_1 := f_1$ and $g_j := f_{n_j} - f_{n_{j-1}}$ for $j \geq 2$ so that

$$\sum_{j=1}^{k} g_j = f_{n_k}$$

and

$$\sum_{j=1}^{\infty} \|g_j\|_p \leq \|f_1\|_p + \sum_{j=1}^{\infty} 2^{-j} = \|f_1\|_p + 1 < \infty.$$

Denote $F_k := \sum_{j=1}^{k} |g_j|$ and $F := \sum_{j=1}^{\infty} |g_j|$. Minkowski's inequality (Theorem 6.3) implies

$$\|F_k\|_p \leq \sum_{j=1}^{k} \|g_j\|_p \leq \|f_1\|_p + 1$$

for all $k \in \mathbb{N}$. For $p<\infty$, the monotone convergence theorem implies

$$\int_X |F|^p \, d\mu = \lim_{k\to\infty} \int_X |F_k|^p \, d\mu \leq (\|f_1\|_p + 1)^p < \infty,$$

so $F \in L^p(X,\mu)$. For $p=\infty$, we observe that

$$\{x \in X : F(x) > \|f_1\|_\infty + 1\} = \bigcup_{k=1}^{\infty} \{x \in X : F_k(x) > \|f_1\|_\infty + 1\}$$

is a μ-null set. So $\|F\|_\infty \leq \|f_1\|_\infty + 1$ and $F \in L^\infty(X,\mu)$. In all cases we see that $F < \infty$ μ-almost everywhere and so

$$h := \sum_{j=1}^{\infty} g_j = \lim_{k\to\infty} \sum_{j=1}^{k} g_j = \lim_{k\to\infty} f_{n_k}$$

exists μ-almost everywhere with $|h| \le F$. Hence $h \in L^p(X,\mu)$. Moreover

$$|h - f_{n_k}|^p \le (F + F_k)^p \le (2F)^p$$

and so for $p<\infty$ the dominated convergence theorem implies

$$\|h - f_{n_k}\|_p^p = \int_X |h - f_{n_k}|^p \, d\mu \longrightarrow 0.$$

For $p=\infty$ and μ-almost every $x \in X$

$$|h(x) - f_{n_k}(x)| \le \sum_{j=k+1}^{\infty} |g_j(x)| \le \sum_{j=k+1}^{\infty} \|g_j\|_\infty .$$

Thus $\|h - f_{n_k}\|_\infty \le \sum_{j=k+1}^{\infty} \|g_j\|_\infty \xrightarrow{k\to\infty} 0$. So in all cases $\|h - f_{n_k}\|_p \to 0$. Since the original sequence was Cauchy, we have $\|h - f_n\|_p \to 0$, and hence $L^p(X,\mu)$ is complete. □

$\boxed{\text{Theorem 6.5}}$ For $1 \le p < \infty$, the set of simple functions $\phi = \sum_{j=1}^{n} \alpha_j 1_{E_j}$ with $|\alpha_j|^p \mu(E_j) < \infty$ for each $j=1,\dots,n$ are dense in $L^p(X,\mu)$. The set of all simple functions is dense in $L^\infty(X,\mu)$.

Proof Note that the assumption $|\alpha_j|^p \mu(E_j) < \infty$ implies $\alpha_j = 0$ if $\mu(E_j) = \infty$. Thus these simple functions belong to $L^p(X,\mu)$ for all $1 \le p \le \infty$. Given $f \in L^p(X,\mu)$, we use Theorem 2.10 to find a sequence of simple functions $(\phi_n)_{n\in\mathbb{N}}$ satisfying $|\phi_1| \le |\phi_2| \le \cdots \le |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set where f is bounded. The latter implies, if $p=\infty$, that $\phi_n \to f$ uniformly on $\{x \in X : |f(x)| \le \|f\|_\infty\}$ which is co-null. Consequently

$$\|\phi_n - f\| = \sup\{ |\phi_n(x) - f(x)| : x \in X,\ |f(x)| \le \|f\|_\infty \} \longrightarrow 0$$

For $p<\infty$, observe that if $\phi_n = \sum_{j=1}^{d} \alpha_j 1_{E_j}$ then for $j=1,\dots,d$.

$$|\alpha_j|^p \mu(E_j) \le \int_X |\phi_n|^p \, d\mu \le \int_X |f|^p \, d\mu < \infty$$

Also, the dominated convergence theorem implies $\|\phi_n - f\|_p \to 0$. □

6.2 The Dual of $L^p(X,\mu)$

Proposition 6.6 Let X be a Banach space. For a linear functional $\varphi: X \to \mathbb{C}$, the following are equivalent:

1. φ is uniformly continuous
2. φ is continuous
3. φ is continuous at $0 \in X$
4. There exists $C > 0$ so that $|\varphi(x)| \le C\|x\|$ for all $x \in X$.

In this case, the infimum of all C satisfying (4) is given by

$$\sup_{x \neq 0} \frac{|\varphi(x)|}{\|x\|} = \sup_{\|x\| \le 1} |\varphi(x)| = \sup_{\|x\|=1} |\varphi(x)|$$

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate.

(3) $\Rightarrow$ (4): For $\varepsilon = 1$, let δ be as in the definition of continuity at $0 \in X$. Then $\|x\| < \delta$ implies $|\varphi(x)| < 1$. For arbitrary $x \in X \setminus \{0\}$, $\|\frac{\delta}{2\|x\|} x\| = \frac{\delta}{2} < \delta$, so

$$\left|\varphi\left(\frac{\delta}{2\|x\|} x\right)\right| < 1 \implies |\varphi(x)| < \frac{2}{\delta}\|x\|$$

(4) $\Rightarrow$ (1): Given $\varepsilon > 0$, set $\delta := \frac{\varepsilon}{C}$. Then $\|x-y\| < \delta = \frac{\varepsilon}{C}$ implies

$$|\varphi(x) - \varphi(y)| = |\varphi(x-y)| \le C\|x-y\| < \varepsilon.$$

For the final claim, denote the suprema by C_1, C_2, and C_3, respectively. If $C > 0$ satisfies $|\varphi(x)| \le C\|x\|$ for all $x \in X$, then $C \ge |\varphi(x)|/\|x\|$ for all $x \neq 0$. Hence $C \ge C_1$, and since $|\varphi(x)| \le C_1\|x\|$ for all $x \in X$ we see that C_1 is indeed this infimum. Finally

$$C_1 \ge \sup_{\|x\| \le 1} \frac{|\varphi(x)|}{\|x\|} \ge C_2 \ge C_3 = \sup_{x \neq 0} \left|\varphi\left(\frac{x}{\|x\|}\right)\right| = C_1$$

So $C_1 = C_2 = C_3$. $\square$

Def For a continuous linear functional $\varphi: X \to \mathbb{C}$ on a Banach space, the quantity in Proposition 6.6 is called the <u>norm</u> of φ and is denoted $\|\varphi\|$. The set of such all continuous linear functionals is called the <u>dual space</u> of X and is denoted X^*. $\square$

Ex (The Riesz Representation Theorem) Let $C_0(\mathbb{R})$ denote the set of continuous functions $f: \mathbb{R} \to \mathbb{C}$ satisfying

$$\lim_{x \to \pm\infty} f(x) = 0.$$

Then $C_0(\mathbb{R})$ is a Banach space with norm

$$\|f\|_u := \sup_{x \in \mathbb{R}} |f(x)|,$$

and $C_0(\mathbb{R})^*$ is isomorphic to the set of regular complex Borel measures on $\mathbb{R}$ equipped the total variation norm. (In fact, this theorem holds more generally for locally compact Hausdorff topological spaces X once one develops the appropriate notions of $C_0(X)$ and regular measures on X.) $\square$

Our main objective in this section is to show for conjugate exponents $1 \le p, q \le \infty$, that $L^p(X,\mu)^* \cong L^q(X,\mu)$ for $p < \infty$, and that this fails for $p = \infty$. First observe Hölder's inequality implies that for

$g \in L^q(X,\mu)$

$$\varphi_g : L^p(X,\mu) \longrightarrow \mathbb{C}$$
$$f \longmapsto \int_X fg\,d\mu$$

defines a linear functional satisfying $|\varphi_g(f)| \le \|fg\|_1 \le \|f\|_p \|g\|_q$. Hence φ_g is continuous by Proposition 6.6 with $\|\varphi_g\| \le \|g\|_q$. In fact, this is typically an equality:

Proposition 6.7 For $1 \le q < \infty$ and $g \in L^q(X,\mu)$, $\|\varphi_g\|_q = \|g\|_q$. If μ is semifinite, then this also holds for $q=\infty$.

Proof Equality is clear for $g=0$ μ-almost everywhere, so assume this is not the case and hence $\|g\|_q > 0$. For $q<\infty$, define

$$f := \frac{|g|^{q-1}\overline{\operatorname{sgn}(g)}}{\|g\|_q^{q-1}} \qquad (f = \overline{\operatorname{sgn}(g)} \text{ for } q=1)$$

Recall $(q-1)p = q$ for conjugate exponents, so

$$\|f\|_p^p = \int_X \frac{|g|^{(q-1)p}}{\|g\|_q^{(q-1)p}}\,d\mu = \frac{1}{\|g\|_q^q}\int_X |g|^q\,d\mu = 1 \qquad (\|f\|_\infty = 1 \text{ for } q=1,\ p=\infty \text{ since } g \neq 0)$$

and

$$\varphi_g(f) = \int_X \frac{|g|^{q-1}\overline{\operatorname{sgn}(g)}}{\|g\|_q^{q-1}}\,g\,d\mu = \frac{1}{\|g\|_q^{q-1}}\int_X |g|^q\,d\mu = \|g\|_q.$$

Hence $\|\varphi_g\| \ge \|g\|_q$.

For μ semifinite and $q=\infty$, let $\varepsilon>0$ and consider

$$E := \{x \in X : |g(x)| \ge \|g\|_\infty - \varepsilon\}.$$

Then there exists measurable $F \subset E$ with $0 < \mu(F) < \infty$. Setting

$$f := \frac{1}{\mu(F)} 1_F \overline{\operatorname{sgn}(g)}$$

we have $\|f\|_1 = 1$ and

$$\varphi_g(f) = \frac{1}{\mu(F)}\int_F |g|\,d\mu \ge \frac{1}{\mu(F)}\int_F \|g\|_\infty - \varepsilon\,d\mu = \|g\|_\infty - \varepsilon.$$

Hence $\|\varphi_g\| \ge \|g\|_\infty - \varepsilon$, and letting $\varepsilon \to 0$ completes the proof. □

This shows the map $L^q(X,\mu) \ni g \mapsto \varphi_g \in L^p(X,\mu)^*$ is injective. Toward showing it is surjective, we require the following lemma:

Lemma 6.8 Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space, and let $1 \le p,q \le \infty$ be conjugate exponents. Denote by Σ the set of simple functions $\phi = \sum_{j=1}^n \alpha_j 1_{E_j}$ satisfying $|\alpha_j|\mu(E_j) < \infty$ for each $j=1,\dots,n$. If an $\mathcal{M}$-measurable function $g : X \to \mathbb{C}$ satisfies $\phi g \in L^1(X,\mu)$ for all $\phi \in \Sigma$ and

$$M_q(g) := \sup\left\{ \left|\int_X \phi g\,d\mu\right| : \phi \in \Sigma \text{ with } \|\phi\|_p = 1 \right\} < \infty,$$

then $g \in L^q(X,\mu)$ with $\|g\|_q = M_q(g)$.

Proof Suppose $f \in L^p(X,\mu)$ is bounded and satisfies $\|f\|_p = 1$ and $E := \{x \in X : f(x) \neq 0\}$ has finite measure. Using Theorem 2.10 we can find simple functions $(\phi_n)_{n\in\mathbb{N}}$ satisfying $|\phi_n| \le |f|$ and

$\phi_n \to f$ μ-almost everywhere. Note that $|\phi_n| \le |f| \le \|f\|_\infty 1_E$ implies $\frac{\phi_n}{\|\phi_n\|_p} \in \Sigma$ and $\|\phi_n\|_p \le \|f\|_p = 1$. Also the hypotheses on g imply

$$|\phi_n g| \le |f g| \le \|f\|_\infty 1_E |g| \in L^1(X,\mu).$$

So the dominated convergence theorem implies

$$\left|\int_X fg\, d\mu\right| = \lim_{n\to\infty} \left|\int_X \phi_n g\right| d\mu \le \limsup_{n\to\infty} \left|\int_X \frac{\phi_n}{\|\phi_n\|_p} g\, d\mu\right| \le M_q(g).$$

Now, suppose $q<\infty$ and let $E_1 \subset E_2 \subset \cdots$ be finite measure sets such that $\bigcup_{n=1}^{\infty} E_n = X$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of simple functions converging to g pointwise with $|\psi_n| \le |g|$. Denote $g_n := \psi_n 1_{E_n}$. Then $g_n \to g$ pointwise and $|g_n| \le |g|$. Define

$$f_n := \frac{|g_n|^{q-1}\, \overline{\operatorname{sgn}(g)}}{\|g_n\|_q^{q-1}}.$$

Then the same computation as in the proof of Proposition 6.7 implies $\|f_n\|_p = 1$. Also

$$\{x\in X : f_n(x) \ne 0\} \subset \{x \in X : g_n(x) \ne 0\} \subset E_n$$

has finite measure. So by our initial observation in the proof we have

$$\int |f_n g|\, d\mu = \int f_n g\, d\mu \le M_q(g).$$

Using Fatou's lemma (Theorem 2.18) we then have

$$\begin{aligned}\|g\|_q &\le \liminf_{n\to\infty} \|g_n\|_q = \liminf_{n\to\infty} \frac{1}{\|g_n\|_q^{q-1}} \int_X |g_n|^q\, d\mu \\ &= \liminf_{n\to\infty} \int |f_n g_n|\, d\mu \le \liminf_{n\to\infty} \int |f_n g|\, d\mu \le M_q(g).\end{aligned}$$

So $\|g\|_q \le M_q(g)$, and in particular $g \in L^q(X,\mu)$. Then Hölder's inequality implies

$$\left|\int_X fg\, d\mu\right| \le \|fg\|_1 \le \|f\|_p \|g\|_q = \|g\|_q$$

for all $f \in \Sigma$ so that $M_q(g) \le \|g\|_q$.

Finally, suppose $q=\infty$. For $\varepsilon>0$, let $E := \{x\in X : |g(x)| > M_\infty(g) + \varepsilon\}$. If $\mu(E)>0$, then the σ-finiteness implies $\exists\, F \subset E$ with $0<\mu(F)<\infty$. Define

$$f := \frac{1}{\mu(F)} 1_F\, \overline{\operatorname{sgn}(g)}$$

Then $\|f\|_1 = 1$ and $\{x\in X : f(x)\ne 0\} = F$ has finite measure. So by our initial observation

$$M_\infty(g) \ge \left|\int_X fg\, d\mu\right| = \frac{1}{\mu(F)} \int_F |g|\, d\mu > \frac{1}{\mu(F)} \int_F M_\infty(g)+\varepsilon\, d\mu = M_\infty(g)+\varepsilon,$$

a contradiction. So $\mu(E)=0$, and hence $\|g\|_\infty \le M_\infty(g)+\varepsilon$. Letting $\varepsilon\to 0$ gives $\|g\|_\infty \le M_\infty(g)$. The reverse inequality again follows from Hölder's inequality. □

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Theorem 6.9 Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space, and let $1\le p,q\le\infty$ be conjugate exponents with $p<\infty$. Then for every $\varphi \in L^p(X,\mu)^*$ there exists $g\in L^q(X,\mu)$ such that $\varphi = \varphi_g$. Hence $L^p(X,\mu)^*$ is isometrically isomorphic to $L^q(X,\mu)$.

Proof We first suppose μ is finite. In this case all simple functions are in $L^p(X,\mu)$. Fix $\varphi \in L^p(X,\mu)^*$ and define $\nu : \mathcal{M} \to \mathbb{C}$ by $\nu(E) := \varphi(1_E)$. Since $1_\emptyset = 0$, $\nu(\emptyset) = 0$.

For a disjoint collection $\{E_n : n\in\mathbb{N}\}\subset\mathcal{M}$ let $E := \bigcup_{n=1}^\infty E_n$, and for each $n\in\mathbb{N}$ let $a_n\in\mathbb{C}$ with $|a_n|\le 1$. Then $(\sum_{n=1}^N a_n 1_{E_n})_{N\in\mathbb{N}}$ is a Cauchy sequence with respect to the $\|\cdot\|_p$-norm. Indeed, for integers $M<N$

$$\Big\| \sum_{n=1}^N a_n 1_{E_n} - \sum_{n=1}^M a_n 1_{E_n}\Big\|_p = \Big\|\sum_{n=M+1}^N a_n 1_{E_n}\Big\|_p \overset{p<\infty}{\le} \Big(\sum_{n=M+1}^N \mu(E_n)\Big)^{1/p} \le \mu\Big(\bigcup_{n=M+1}^\infty E_n\Big)^{1/p} \xrightarrow{M\to\infty} 0$$

by continuity from above. Thus these partial sums converge to some $f\in L^p(X,\mu)$ by Theorem 6.4. If we choose $a_n := \overline{\operatorname{sgn}\nu(E_n)}$, then the continuity of φ implies

$$\sum_{n=1}^\infty |\nu(E_n)| = \lim_{N\to\infty}\sum_{n=1}^N a_n\nu(E_n) = \lim_{N\to\infty}\varphi\Big(\sum_{n=1}^N a_n 1_{E_n}\Big) = \varphi(f) < \infty.$$

If we choose $a_n = 1$, then f must equal the pointwise limit of 1_E and hence

$$\sum_{n=1}^\infty \nu(E_n) = \lim_{N\to\infty}\varphi\Big(\sum_{n=1}^N 1_{E_n}\Big) = \varphi(1_E) = \nu(E).$$

So ν is a complex measure on $(X,\mathcal{M})$. Moreover, $\nu \ll \mu$ since $\mu(E)=0$ implies 1_E is identified with $0\in L^p(X,\mu)$ and hence $\nu(E) = \varphi(1_E) = 0$. By the Radon–Nikodym theorem (Theorem 3.13) there exists $g\in L^1(X,\mu)$ satisfying

$$\varphi(1_E) = \nu(E) = \int_E g\,d\mu = \int_X 1_E g\,d\mu.$$

Thus $\varphi(f) = \int_X fg\,d\mu$ for all simple functions f, and

$$\Big|\int_X fg\,d\mu\Big| = |\varphi(f)| \le \|\varphi\|\cdot\|f\|_p.$$

Lemma 6.8 therefore implies $g\in L^q(X,\mu)$. Using the density of simple functions in $L^p(X,\mu)$ from Theorem 6.5, we obtain $\varphi(f) = \int_X fg\,d\mu = \varphi_g(f)$ for all $f\in L^p(X,\mu)$.

Now suppose μ is σ-finite, and let $E_1\subset E_2\subset\cdots\subset X$ be finite measure sets with $\bigcup_{n=1}^\infty E_n = X$. Identify

$$L^p(E_n,\mu) = 1_{E_n}L^p(X,\mu)\subset L^p(X,\mu)$$

Restricting φ to this subspace and applying the first part of the proof yields

$$g_n\in L^q(E_n,\mu) = 1_{E_n}L^q(X,\mu)\subset L^q(X,\mu)$$

such that for all $f\in L^p(E_n,\mu)$

$$\varphi(f) = \int_{E_n} f g_n\,d\mu = \int_X fg\,d\mu.$$

Proposition 6.7 implies

$$\|g_n\|_q = \|\varphi|_{L^p(E_n,\mu)}\| \le \|\varphi\|.$$

Now, for integers $m<n$, since $L^p(E_m,\mu)\subset L^p(E_n,\mu)$ we have

$$\int_{E_m} f g_m\,d\mu = \varphi(f) = \int_{E_n} f g_n\,d\mu = \int_{E_m} f g_n\,d\mu$$

for all $f\in L^p(E_m,\mu)$. It follows that $g_m = g_n$ μ-almost everywhere on E_m (Exercise: check this). Thus we can define $g: X\to\mathbb{C}$ by letting $g = g_n$ μ-almost everywhere on E_n. Note that $|g| = \sup_n |g_n|$ so $\|g\|_q = \lim_{n\to\infty}\|g_n\|_q \le \|\varphi\|$ either by the monotone convergence theorem for $q<\infty$, or by definition of the norm for $q=\infty$. In either case we have $g\in L^q(X,\mu)$. Moreover, for $f\in L^p(X,\mu)$ we have $1_{E_n}f\to f$

in $\|\cdot\|_p$-norm by the dominated convergence theorem. Hence

$$\varphi(f) = \lim_{n\to\infty} \varphi(1_{E_n} f) = \lim_{n\to\infty} \int_X 1_{E_n} f g_n \, d\mu = \lim_{n\to\infty} \int_X 1_{E_n} f g \, d\mu = \int_X f g \, d\mu,$$

where the last equality again uses the dominated convergence theorem. Thus $\varphi = \varphi_g$. □

For $1<p<\infty$ and $(X,\mathcal{M},\mu)$ σ-finite, the previous theorem implies

$$\left(L^p(X,\mu)^*\right)^* \cong L^q(X,\mu)^* \cong L^p(X,\mu)$$

where q is the conjugate exponent to p. We say $L^p(X,\mu)$ is reflexive.

For $p=2$ one has $q=2$, and so in this case we have $L^2(X,\mu)^* \cong L^2(X,\mu)$. In fact, $L^2(X,\mu)$ is a Hilbert space with inner product

$$\langle f,g\rangle_2 := \int_X f\bar{g}\, d\mu \qquad f,g\in L^2(X,\mu).$$

When $p\neq 2$, one can prove $\|\cdot\|_p$ does not come from an inner product.

Finally, we used $p<\infty$ in the above proof to show the measure ν was countably additive. The following example shows there is no way to patch the $p=\infty$ case:

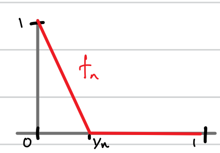
EX Consider $([0,1],\mathcal{L},m)$. Observe that for continuous $f:[0,1]\to\mathbb{C}$

$$|f(0)| \le \sup_{0\le t\le 1} |f(t)| = \|f\|_\infty.$$

Thus $\varphi(f)=f(0)$ defines a norm 1 linear functional on the set of continuous functions $C[0,1]\subset L^\infty([0,1],m)$. A result from functional analysis called the Hahn-Banach theorem (which is proven via Zorn's Lemma) implies φ can be extended to a linear functional on $L^\infty([0,1],m)$ still with norm 1. Denote this extension by φ again. Suppose, towards a contradiction, that there exists $g\in L^1([0,1],m)$ such that

$$\varphi(f) = \int_{[0,1]} fg\, dm \qquad f\in L^\infty([0,1],m).$$

For each $n\in\mathbb{N}$, consider $f_n = \max(1-nx,0)\in C[0,1]$. Then $\varphi(f_n)=f_n(0)=1$ for all $n\in\mathbb{N}$, but $f_n\to 0$ m-almost everywhere. Since $|f_n g|\le |g|\in L^1([0,1],m)$, the dominated convergence theorem implies



$$1 = \lim_{n\to\infty} \varphi(f_n) = \lim_{n\to\infty} \int_{[0,1]} f_n g\, d\mu = 0,$$

a contradiction. □

Despite the above, $L^\infty(X,\mu)$ does have a major advantage over the other L^p spaces: it is a $*$-algebra. In fact, it is an example of a von Neumann algebra. But that is a story for another time...

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