Nets

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Roughly speaking, nets are a generalization of sequences wherein the indexing set $\mathbb{N}$ is replaced by a *directed set*. As the name suggests, these sets have a notion of direction much like $\mathbb{N}$ does $(1 \to 2 \to 3 \cdots)$, however they may be uncountable and may have multiple paths to "infinity." The elements that are indexed by a directed set live in a topological space so that one can consider the notion of convergence of a net. Net are essential for general topology in the sense that they can characterize closedness, compactness, and continuity in the same way that sequences do in metric spaces.

1 Directed Sets

Definition 1.1. A **directed set** I is a set equipped with a binary relation $\leq$ that satisfies:

(i) $i \leq i$ for all $i \in I$ (reflexive);

(ii) if $i \leq j$ and $j \leq k$, then $i \leq k$ (transitive);

(iii) for any $i, j \in I$ there exists $k \in I$ with $i, j \leq k$ (upper bound property).

Typically reflexivity and transitivity are obvious, whereas the upper bound property may need to be justified.

Example 1.2. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are all directed sets with the usual ordering. In fact, any subset of $\mathbb{R}$ (even finite ones) are directed sets with the order they inherit from $\mathbb{R}$.

Example 1.3. Let X be a set, and let $\mathscr{F}$ denote the collection of all finite subsets of X. For $A, B \in \mathscr{F}$, write $A \leq B$ if $A \subset B$. This makes $\mathscr{F}$ into a directed set. Note that $A \cup B$ serves as an upper bound for both A and B.

Example 1.4. Let X be a topological space, and fix $x_0 \in X$. Let $\mathscr{N}(x_0)$ denote the collection of open neighborhoods of x_0. For $A, B \in \mathscr{N}(x_0)$, write $A \leq B$ if $A \supset B$. This makes $\mathscr{N}(x_0)$ into a directed set where $A \cap B$ is an upper bound for A and B.

Example 1.5. Let X be a topological space. Then $\{(\epsilon, K) \colon \epsilon > 0\ K \subset X \text{ compact}\}$ is a directed set where

$$(\epsilon, K) \leq (\epsilon', K')$$

if and only if $\epsilon \geq \epsilon'$ and $K \subset K'$. (**Exercise:** determine a common upper bound for (ϵ, K) and (ϵ', K').)

2 Nets

Definition 2.1. Let X be a topological space. A **net** in X is a map $x \colon I \to X$ where I is a directed set.

A net $x \colon I \to X$ is usually denoted $(x(i))_{i \in I}$ or $(x_i)_{i \in I}$ where $x_i := x(i)$. This is supposed to remind you of sequence notation. As with sequences in a metric space, there is a notion of convergence:

Definition 2.2. A net $(x_i)_{i \in I}$ **converges** to $x \in X$ if for every open subset $U \subset X$ containing x there is $i_0 \in I$ so that $x_i \in U$ whenever $i \geq i_0$. In this case we call x the **limit** of the net and write

$$x = \lim_i x_i.$$

When $I = \mathbb{N}$, this is simply the usual notion of convergence for a sequence. When $I = \mathbb{R}$ this is also capturing familiar behavior:

Example 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Recall that we say f has a limit at ∞ if there exists $L \in \mathbb{R}$ so that for all $\epsilon > 0$ there exists $t_0 \in \mathbb{R}$ so that

$$|f(t) - L| < \epsilon \qquad \forall t \geq t_0.$$

But this is precisely saying that the net $(f(t))_{t\in\mathbb{R}}$ converges to L.

Example 2.4. Let X be a topological space, let $x_0 \in X$ and let $\mathscr{N}(x_0)$ be as in Example 1.4. For each $U \in \mathscr{N}(x_0)$ pick any point in U and label it x_U. Then $(x_U)_{U\in\mathscr{N}(x_0)}$ is a net which converges to x_0. Indeed, let $U \subset X$ be an open set containing x_0. Then $U \in \mathscr{N}(x_0)$ and for any $U' \in \mathscr{N}(x_0)$ with $U' \geq U$, we have $x_{U'} \in U' \subset U$.

Example 2.5. Let X be a topological space and let $f : X \to \mathbb{C}$ be a function. For each pair (ϵ, K) as in Example 1.5, let $f_{(\epsilon,K)}$ be any function $g : X \to \mathbb{C}$ satisfying $|f(x) - g(x)| < \epsilon$ for all $x \in K$. Then the net $(f_{(\epsilon,K)})$ converges to f in the topology of uniform convergence on compact subsets. Indeed, fix $K \subset X$ compact. Let $\epsilon > 0$, then for any $(\epsilon', K') \geq (\epsilon, K)$ we have $|f(x) - f_{(\epsilon',K')}| < \epsilon' \leq \epsilon$ for all $x \in K'$; in particular, for all $x \in K$.

Proposition 2.6. *Let X be a topological space. Then $V \subset X$ is closed if and only if for every convergent net $(x_i)_{i\in I} \subset V$ one has* $\lim_i x_i \in V$.

Proof. $(\Rightarrow)$: Let $(x_i)_{i\in I} \subset V$ be a convergent net. Suppose, towards, a contradiction, that $x := \lim_i x_i$ is not contained in V. Then $x \in V^c$ which is an open set. Consequently, by definition of the convergence of a net, there exists $i_0 \in I$ such that $x_i \in V^c$ for all $i \geq i_0$. But this contradicts $x_i \in V$ for all $i \in I$. Thus it must be that $x \in V$.
$(\Leftarrow)$: To show that V is closed, we will show that V^c is open. Suppose, towards a contradiction, that there exists $x \in V^c$ such that for all open subsets U containing x one has $U \cap V \neq \emptyset$. Let $\mathscr{N}(x)$ be as in Example 1.4. For each $U \in \mathscr{N}(x)$, let $x_U \in U \cap V$. Then $(x_U)_{U\in\mathscr{N}(x)} \subset V$ and it converges to x by Example 2.4. By assumption we must have $x \in V$, but this contradicts $x \in V^c$. Thus for any $x \in V^c$ there is an open set containing which does not intersect V; that is, V^c is open. ☐

We say a subset $S \subset X$ in a topological space is *sequentially closed* if whenever $(x_n)_{n\in\mathbb{N}} \subset S$ is a convergent sequence one has $\lim_n x_n \in S$. Since sequences are particular kinds of nets, the above proposition implies that closed sets are sequentially closed. In a metric space, the two notions are equivalent. However, for general topological spaces sequentially closed does not imply closed, as the following example illustrates.

Example 2.7. [1] Consider $\mathbb{R}^{\mathbb{R}}$ with the product topology, which we think of as arbitrary functions $f : \mathbb{R} \to \mathbb{R}$. Recall that under the product topology, an open subset of $\mathbb{R}^{\mathbb{R}}$ is a union of subsets of the form

$$\prod_{t\in\mathbb{R}} U_t,$$

where $U_t \subset \mathbb{R}$ is open for all $t \in \mathbb{R}$ and $U_t \neq \mathbb{R}$ for only finitely many $t \in \mathbb{R}$. Consequently, a net $(f_i)_{i\in I} \subset \mathbb{R}^{\mathbb{R}}$ converges to $f \in \mathbb{R}^{\mathbb{R}}$ if and only if they converge pointwise as functions on $\mathbb{R}$. Let B be the subset of Borel functions. Then B is sequentially closed because we know from measure theory that the pointwise limit of a sequence of Borel functions is Borel. B is also also dense. Indeed, let $f \in \mathbb{R}^{\mathbb{R}}$. Let $\mathscr{F}$ be the collection of finite subsets of $\mathbb{R}$, ordered by inclusion. Then for each $F \in \mathscr{F}$ we can find a polynomial p_F such that $p_F(t) = f(t)$ for each $t \in F$. The net $(p_F)_{F\in\mathscr{F}}$ converges pointwise to f and consists of Borel functions. Therefore the closure of B is all of $\mathbb{R}^{\mathbb{R}}$. On the otherhand, we know there are non-Borel functions so B is not closed.

Proposition 2.8. *Let X and Y be topological spaces. Then $f : X \to Y$ is continuous if and only if for every convergent net $(x_i)_{i\in I} \subset X$ one has that $(f(x_i))_{i\in I} \subset Y$ is a convergent net with* $\lim_i f(x_i) = f(\lim_i x_i)$.

[1] Thanks to Ben Hayes for supplying this example.

Proof. (⇒): Suppose f is continuous and $(x_i)_{i\in I} \subset X$ converges to some $x \in X$. Let $U \subset Y$ be an open subset containing $f(x)$. Then $f^{-1}(U) \subset X$ is an open subset containing x. Consequently there exists $i_0 \in I$ such that for all $i \geq i_0$ we have $x_i \in f^{-1}(U)$. Thus for all $i \geq i_0$ we have $f(x_i) \in U$. So $(f(x_i)_{i\in I})$ converges to $f(x)$.
(⇐): Let $U \subset Y$ be an open subset. We must show $f^{-1}(U)$ is open. If not, then there is an $x \in f^{-1}(U)$ such that $N \cap f^{-1}(U)^c \neq \emptyset$ for all $N \in \mathscr{N}(x)$. We can then define a net by letting $x_N \in N \cap f^{-1}(U)^c$ for each $N \in \mathscr{N}(x)$. Then the net $(x_N)_{N\in\mathscr{N}(x)}$ converges to x by Example 2.4. By construction, $f(x_N) \in U^c$ for all $N \in \mathscr{N}(x)$. By assumption, $(f(x_N))_{N\in\mathscr{N}(x)}$ converges to $f(x)$, and since U^c is closed the previous proposition implies $f(x) \in U^c$. But this contradicts $x \in f^{-1}(U)$. Thus $f^{-1}(U)$ must be open and therefore f is continuous. □

Let (X, d) be a metric space. We say a net $(x_i)_{i\in I} \subset X$ is *Cauchy* if for all $\epsilon > 0$ there exists $i_0 \in I$ so that whenver $i, j \geq i_0$ we have $d(x_i, x_j) < \epsilon$. We conclude this section by examining Cauchy nets in a metric spaces. In particular, we will show that Cauchy nets in a complete metric space converge. The idea is to extract a Cauchy sequence from the Cauchy net, so as to use the completeness.

Proposition 2.9. *Let (X, d) be a complete metric space and let $(x_i)_{i\in I}$ be a Cauchy net. Then $(x_i)_{\in I}$ converges.*

Proof. Let $i(1) \in I$ be such that $d(x_i, x_j) < 1$. Let $i(2) \in I$ be such that $i(2) \geq i(1)$ and $d(x_i, x_j) < \frac{1}{2}$ for all $i, j \geq i(2)$. We inductively find $i(n) \in I$ for each $n \in \mathbb{N}$ such that $i(n) \geq i(n-1)$ and $d(x_i, x_j) < \frac{1}{n}$ for all $i, j \geq i(n)$. We claim that the sequence $(x_{i(n)})_{n\in\mathbb{N}}$ is Cauchy. Indeed, let $\epsilon > 0$. If $N \in \mathbb{N}$ satisfies $\frac{1}{N} < \epsilon$, then for $n, m \geq N$ we have $d(x_{i(n)}, x_{i(m)}) < \frac{1}{N} < \epsilon$. Since (X, d) is complete, $(x_{i(n)})_{n\in\mathbb{N}}$ converges to some $x \in X$. We claim the original net also converges to this x. Indeed, let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x_{i(n)}, x) < \frac{\epsilon}{2}$. By choosing a larger N if necessary, we may assume $\frac{1}{N} \leq \frac{\epsilon}{2}$. Then for any $i \geq i(N)$ we have

$$d(x_i, x) \leq d(x_i, x_{i(N)}) + d(x_{i(N)}, x) < \frac{1}{N} + \frac{\epsilon}{2} \leq \epsilon.$$

Hence the $(x_i)_{i\in I}$ converges to x. □

Remark 2.10. When (X, d) is a metric space, any Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is bounded. Indeed, let $N \in \mathbb{N}$ be such that $d(x_n, x_m) \leq 1$ for all $n, m \geq N$. Then setting $R := \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}$, we have $(x_n)_{n\in\mathbb{N}} \subset B(x_N, R)$. This same argument does **not** work for nets. We can still find $i_0 \in I$ such that $d(x_i, x_j) \leq 1$ for all $i, j \geq i_0$, but then there are not necessarily finitely many $i \leq i_0$. For example, the net $(e^{-t})_{t\in\mathbb{R}}$ converges in $\mathbb{R}$ to zero but is not bounded.

3 Subnets

Subnets are the analogue of subsequences, though they are a bit more subtle.

Definition 3.1. Let $(x_i)_{i\in I}$ be a net in a topological space. Then $(y_j)_{j\in J}$ is a **subnet** of $(x_i)_{i\in I}$ if there exists a map $\sigma\colon J \to I$ such that

(i) $x_{\sigma(j)} = y_j$ for all $j \in J$;

(ii) if $j_1 \leq j_2$ then $\sigma(j_1) \leq \sigma(j_2)$ (monotone);

(iii) for any $i \in I$ there exists $j \in J$ such that $\sigma(j) \geq i$ (final).

Example 3.2. For a sequence $(x_n)_{n\in\mathbb{N}}$, any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is a subnet where $\sigma(k) = n_k$. However, because we only require the map σ to be monotone (rather than strictly monotone) there are subnets of the sequence which are **not** subsequences. For example, $(x_1, x_1, x_1, x_2, x_3, \ldots)$ is a valid subnet, even though it is not a vallid subsequence.

Proposition 3.3. *Let X be a topological space. If a net $(x_i)_{i\in I} \subset X$ converges, then every subnet converges to the same limit.*

Proof. Let $x := \lim_i x_i$. Let $(y_j)_{j\in J}$ be a subnet with monotone final map $\sigma\colon J \to I$. Let $U \subset X$ be an open subset containing x. Then there exists $i_0 \in I$ such that $x_i \in U$ for all $i \geq i_0$. By finality there exists $j_0 \in J$ such that $\sigma(j_0) \geq i_0$. Thus by monotonicity we for all $j \geq j_0$ that $\sigma(j) \geq \sigma(j_0) \geq i_0$ and hence $y_j = x_{\sigma(j)} \in U$. That is, $(y_j)_{j\in J}$ converges to x. □

Finally, we conclude this note by characterizing compactness in terms of convergent subnets. This is the analogue of the fact that in a metric space a set is compact if and only if every sequence in it has a convergent subsequence (which is sometimes called being sequentially compact).

Proposition 3.4. *Let X be a topological space. Then $K \subset X$ is compact if and only if every net $(x_i))_{i\in I} \subset K$ has a convergent subnet.*

Proof. ($\Rightarrow$): Let K be a compact. We recall that it has the finite intersection property: if $\{C_i\}_{i\in I}$ is a collection of closed subsets of K satisfying $\bigcap_{i\in F} C_i \neq \emptyset$ for any finite subset $F \subset I$, then $\bigcap_{i\in I} C_i \neq \emptyset$. Indeed, otherwise $\{C_i^c\}_{i\in I}$ is an open cover for K with no finite subcover.

Now, let $(x_i)_{i\in I} \subset K$ be a net. Define $C_i := \overline{\{x_j : j \geq i\}}$. Then for $F \subset I$ finite, we can find $j \geq i$ for each $i \in F$ and so

$$x_j \in \bigcap_{i\in F} C_i \neq \emptyset$$

By the finite intersection property we therefore have $\bigcap_{i\in I} C_i \neq \emptyset$. Let y be an element of this set. Then for every $i \in I$, $y \in C_i$ which means for every neighborhood U of y, $U \cap \{x_j : j \geq i\} \neq \emptyset$. That is, for every $i \in I$ and every neighborhood U, there exists $j \geq i$ such that $x_j \in U$. Set $y_{(U,j)} := x_j$. Then $(y_{(U,j)})$ is a net (where $(U, j) \leq (U', j')$ means $U \supset U'$ and $j \leq j'$), which converges to y. Defining $\sigma(U, j) := j$ yields a monotone final map and so $(y_{(U,j)})$ is a (convergent) subnet of $(x_i)_{i\in I}$.

($\Leftarrow$): Towards a contradiction, let $\{U_i : i \in I\}$ be an open cover of K with no finite subcover. Let $\mathscr{F}$ be the collection of finite subsets of I, which we make into a directed set by ordering by inclusion. For each $F \in \mathscr{F}$ let x_F be any point in $K \setminus \bigcup_{i\in F} U_i$ (which exists by virtue of there being no finite subcover). Then $(x_F)_{F\in\mathscr{F}}$ is a net and consequently has a convergent subnet $(x_{\sigma(j)})_{j\in J}$, say with limit x. Then $x \in U_i$ for some $i \in I$ and consequently there is $j_0 \in J$ such that $x_{\sigma(j)} \in U_i$ for all $j \geq j_0$. Let $j_1 \in J$ be such that $\sigma(j_1) \geq \{i\} \in \mathscr{F}$. Then there exists $j \geq j_1$ and $j \geq j_0$. For this j we have $x_{\sigma(j)} \in U_i$ but $\sigma(j) \geq \sigma(j_1) \geq \{i\}$ implies $x_{\sigma(j)} \notin U_i$, a contradiction. Thus every open cover of K has a finite subcover and K is therefore compact. □