Math 104 - Introduction to Analysis (MWF, 10-11am)

8/23/2017 • Info: Brent Nelson

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https://math.berkeley.edu/~brent/104.html

(1) "know" $\xrightarrow{104}$ know (2) Analysis proofs (3) Functional (via Metric Spaces)

The Real Numbers

• We begin with an examination of $\mathbb{R}$, the real numbers

Contains:
- $\mathbb{N}$ natural numbers $\{1,2,3,\ldots\}$
- in $\mathbb{Z}$ integers $\{\ldots,-2,-1,0,1,2,\ldots\}$
- in $\mathbb{Q}$ rational numbers $\{\frac{n}{m};\ n,m\in\mathbb{Z}\ m\neq 0\}$
- $\mathbb{R}\setminus\mathbb{Q}$ irrational numbers.

informally:
$\mathbb{R}$ is the set of #'s you're used to dealing with in science classes.

• Formally, there are two ways to obtain $\mathbb{R}$:

verbal

(1) Construct them from $\mathbb{Q}$ using "Dedekind cuts" or as "completions of Cauchy sequences".
Downsides: involved and still have to show it has all the nice properties you want.

(2) Define $\mathbb{R}$ as a set w/ all the nice properties you want

Def We define the real numbers $\mathbb{R}$ to be a set equipped with two operations:

addition: $\mathbb{R}\times\mathbb{R}\ni(a,b)\longmapsto$ "$a+b$" $\in\mathbb{R}$

multiplication: $\mathbb{R}\times\mathbb{R}\ni(a,b)\longmapsto$ "$a\cdot b$" $\in\mathbb{R}$

and satisfying the following seven properties

I For every $a,b\in\mathbb{R}$, $a+b=b+a$ and $ab=b\cdot a$ (commutativity)

II For every $a,b,c\in\mathbb{R}$, $(a+b)+c=a+(b+c)$ and $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ (associativity)

III For every $a, b, c \in \mathbb{R}$, $a \cdot (b+c) = a \cdot b + a \cdot c$ (distributivity)

IV $\exists$ distinct elements $0, 1 \in \mathbb{R}$ s.t. $\forall a \in \mathbb{R}$
$a + 0 = a$ and $a \cdot 1 = a$ (neutral/identity elmts)

V For any $a \in \mathbb{R}$, there exists an element $b \in \mathbb{R}$ s.t. $a + b = 0$. Denote $b = -a$.
Also, if $a \neq 0$ then there exists $c \in \mathbb{R}$ s.t. $a \cdot c = 1$, denoted $c = a^{-1} = \frac{1}{a}$
(additive, multiplicative inverses)

VI There is a subset $\mathbb{R}_+$ of $\mathbb{R}$ s.t.
(1) If $a, b \in \mathbb{R}_+$, then $a+b, a \cdot b \in \mathbb{R}_+$
(2) For any $a \in \mathbb{R}$, exactly one of the following holds:
$a \in \mathbb{R}_+$,
$a = 0$, or
$-a \in \mathbb{R}_+$ (ordered)

VII "least upper bound property" (later)

Properties I–V make $\mathbb{R}$ into a "field", and are properties you're familiar w/ from arithmetic. But even with just these properties we can start to prove things about $\mathbb{R}$:

Lemma For $a, b \in \mathbb{R}$, there is exactly one $x \in \mathbb{R}$ s.t. $x + a = b$.

Pf: First note that $b + -a$ is a solution.
Now, suppose x is another solution. Then
$$x \underset{\text{IV}}{=} x + 0 \underset{\text{V}}{=} x + (a + -a) \underset{\text{II}}{=} (x+a) + -a = b + -a.$$

Explain that you want this level of detail on 1st HW

So every solution is equal to $b + -a$; that is it is the only solution. $\square$

- Corollary: The additive identity $0 \in \mathbb{R}$ is unique
  Pf: Suppose $0' \in \mathbb{R}$ is another neutral additive element; that is, $a + 0' = a$ for all $a \in \mathbb{R}$. But since $x + a = a$ has a unique solution by our lemma, we must have $0 = 0'$. $\square$

- Prop^{tion} For any $a \in \mathbb{R}$, $a \cdot 0 = 0$
  Pf: Let $a \in \mathbb{R}$, then
  $$a \cdot 0 + a \cdot 0 \underset{\text{III}}{=} a \cdot (0 + 0) \underset{\text{IV}}{=} a \cdot (0)$$
  But also, $0 + a \cdot 0 \underset{\text{IV}}{=} a \cdot 0$. So by lemma $a \cdot 0 = 0$ since both are solutions to $x + a.0 = a.0$. $\square$

- Using similar, meticulous arguments we can prove lots of other familiar arithmetic facts (see F1 – F10 in book).

Prop: For $a, b \in \mathbb{R}$

(i) If $a \neq 0$, $xa = b$ has a unique solution.
(ii) $-(-a) = a$
(iii) $(a^{-1})^{-1} = a$
(iv) $-(a+b) = -a + -b$
(v) $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$
(vi) $(-1) \cdot a = -a$

Pf Exercise. $\square$

Order Property: VI

- Def: $\mathbb{R}_+$ is called the set of positive numbers. The set of $a \in \mathbb{R}$ s.t. $-a \in \mathbb{R}_+$ is called the set of negative numbers and is denoted $\mathbb{R}_-$