Proposition: Let S be a non-empty, closed subset of $\mathbb{R}$. If S is bounded from above, then $\sup(S) \in S$. If S is bounded from below, then $\inf(S) \in S$

Pf: Suppose is bounded from above. Then $\sup(S)$ exists. Suppose, towards a contradiction, that $\sup(S) \notin S$. Then $\sup(S) \in S^c$, and we note that S^c is open since S is closed. Thus $\exists r > 0$ s.t. $B(\sup(S), r) \subseteq S^c$. However, one of the first things we proved about supremums was that we can always find $s \in S$ s.t.

$$\sup(S) - r < s \leq \sup(S).$$

This means $|s - \sup(S)| < r \Rightarrow s \in B(\sup(S), r)$, which contradicts $B(\sup(S), r) \subseteq S^c$. Thus, we must have $\sup(S) \in S$.

Next, if S is bounded from below, then $-S$ is bounded from above. Applying the preceeding argument to $-S$ we have

$$\sup(-S) \in -S.$$

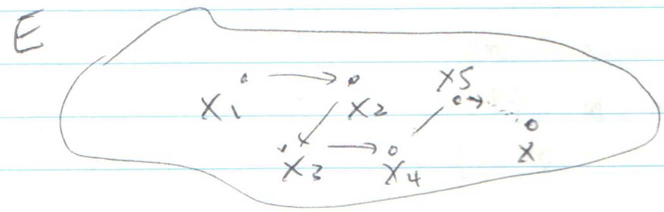
From Homework, we have $\sup(-S) = -\inf(S)$ and so $-\inf(S) \in -S \Rightarrow \inf(S) \in S$. $\square$

Convergent Sequences III.3

Def: In a metric space (E, d) a sequence is an "infinite" ordered list of points in E: $(x_n)_{n \in \mathbb{N}} = (x_1, x_2, x_3, \ldots)$ for $x_n \in E$.

We want to formalize the notion of a limit. That is, given a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$

we want to define what it means for the x_n to ~~[illegible]~~ "tend to" to some fixed $x \in E$.



In particular, it is ok for the first few x_n's to wander about and not approach x, but eventually we want the x_n to be close to x.

Def: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in a metric space (E, d). A point $x \in E$ is called a limit of the sequence $(x_n)_{n \in \mathbb{N}}$ if

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \ d(x_n, x) < \varepsilon.$$

When a sequence has a limit, we say it is convergent. More precisely, if x is a limit, we say the sequence converges to x, and write

$$\lim_{n \to \infty} x_n = x.$$

- Informally, this def. says x is a limit of $(x_n)_{n \in \mathbb{N}}$ if no matter how small we make $\varepsilon > 0$, eventually the whole sequence after some $N \in \mathbb{N}$ is at most a distance of ε away from x.

Ex (1) In $\mathbb{R}$ w/ usual metric, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n = \frac{1}{n} \qquad n \in \mathbb{N}$$

converges 0.

Pf: Let $\varepsilon > 0$. ~~Let~~ Then let $N \in \mathbb{N}$ be s.t. $\frac{1}{N} < \varepsilon$. If $n \geq N$, then

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Thus for all $n \geq N$

$$d(x_n, 0) = d(\tfrac{1}{n}, 0) = |\tfrac{1}{n} - 0| = \tfrac{1}{n} < \varepsilon. \quad \square$$

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(2) In $\mathbb{R}$ w/ usual metric, the sequence

$$(\sqrt{2+\tfrac{1}{n}})_{n\in\mathbb{N}}$$

converges to $\sqrt{2}$.

~~Pf~~ $\boxed{\text{Scratch work}}$ First, let us examine:

$$d(\sqrt{2+\tfrac{1}{n}}, \sqrt{2}) = |\sqrt{2+\tfrac{1}{n}} - \sqrt{2}| = \sqrt{2+\tfrac{1}{n}} - \sqrt{2} \overset{?}{<} \varepsilon$$

Let's "solve" for N ~~which should be when the above quantity is approximately ε~~

$$\sqrt{2+\tfrac{1}{N}} - \sqrt{2} < \varepsilon$$

$$\sqrt{2+\tfrac{1}{N}} < \varepsilon + \sqrt{2}$$

$$2 + \tfrac{1}{N} < \varepsilon^2 + 2\sqrt{2}\cdot\varepsilon + 2$$

$$\tfrac{1}{N} < \varepsilon^2 + 2\sqrt{2}\cdot\varepsilon$$

$$N > \frac{1}{\varepsilon^2 + 2\sqrt{2}\cdot\varepsilon}$$

This tells us we should take $N > \frac{1}{\varepsilon^2 + 2\sqrt{2}\cdot\varepsilon}$ so that

$$\frac{1}{N} < \varepsilon^2 + 2\sqrt{2}\cdot\varepsilon \rightsquigarrow \sqrt{2+\tfrac{1}{N}} - \sqrt{2} < \varepsilon$$

We now proceed with the actual proof:

Pf: Let $\varepsilon > 0$. Take $N > \frac{1}{\varepsilon^2 + 2\sqrt{2}\cdot\varepsilon}$.

Then for $n \geq N$ we have:

$$\frac{1}{n} \le \frac{1}{N} < \varepsilon^2 + 2\sqrt{2}\cdot\varepsilon$$

Thus for $n \ge N$ we have

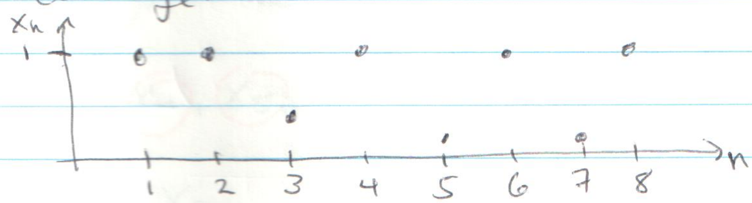
$$d(\sqrt{2+\tfrac{1}{n}}, \sqrt{2}) = |\sqrt{2+\tfrac{1}{n}} - \sqrt{2}| = \sqrt{2+\tfrac{1}{n}} - \sqrt{2}$$
$$< \sqrt{2+\varepsilon^2+2\cdot\sqrt{2}\cdot\varepsilon} - \sqrt{2}$$
$$(\text{FOIL}) = \sqrt{(\varepsilon+\sqrt{2})^2} - \sqrt{2}$$
$$= \varepsilon + \sqrt{2} - \sqrt{2} = \varepsilon. \qquad \square$$

(3) In $\mathbb{R}$ with usual metric, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases}$$

does not converge.

Observe:



Pf: We have to show the sequence has no limit. That is, for every $x \in \mathbb{R}$ we need to show $(x_n)_{n\in\mathbb{N}}$ fails to converge to x.

Let $x \in \mathbb{R}$. To be precise, we must show

(negation) $\neg(\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall n \ge N\ d(x_n, x) < \varepsilon)$

$\rightsquigarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \ge N \text{ s.t. } d(x_n, x) \ge \varepsilon$

Case 1: If $x = 1$, let $\varepsilon = 1/2$. So for any $N \in \mathbb{N}$ we let $n = 2N+1 \ge N$ so that

$$d(x_n, x) = d(\tfrac{1}{2N+1}, 1) = |\tfrac{1}{2N+1} - 1|$$
$$= 1 - \tfrac{1}{2N+1} \ge 1 - \tfrac{1}{2} = \tfrac{1}{2} = \varepsilon.$$

Case 2: If $x \ne 1$, let $\varepsilon = |x-1|$. Then for any $N \in \mathbb{N}$

letting $n = 2N \geq N$ we have
$$d(x_n, x) = d(1, x) = |1-x| = \varepsilon. \quad \square$$

~~Thm~~

<u>Def</u>: For a strictly increasing sequence of natural numbers
$$\mathbb{N} \ni n_1 < n_2 < n_3 < \cdots \quad (\star)$$
and $(x_n)_{n \in \mathbb{N}}$ a sequence in a metric space (E, d), the sequence
$$(x_{n_k})_{k \in \mathbb{N}}$$
is called a subsequence. That is, $(x_{n_k})_{k \in \mathbb{N}}$ is the sequence $(y_k)_{k \in \mathbb{N}}$ defined by: $y_k := x_{n_k}$ for $k \in \mathbb{N}$.

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8$$

$$x_2, x_4, x_5, x_8, \ldots$$
corresponds to $n_1 = 2,\ n_2 = 4,\ n_3 = 5,\ n_4 = 8, \ldots$

EX in ~~Ex~~(3) from before if $n_k := 2k$, then $x_{n_1}, x_{n_2}, \ldots = x_2, x_4, x_6, \ldots = 1, 1, 1, \ldots$

<u>Prop</u>: Let (E, d) be a metric space and let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence. Then
(i) $(x_n)_{n \in \mathbb{N}}$ has exactly one limit
(ii) Any subsequence $(x_{n_k})_{k \in \mathbb{N}}$ converges to the same limit as $(x_n)_{n \in \mathbb{N}}$.
(iii) ~~[illegible]~~ The set $\{x_n : n \in \mathbb{N}\}$ is bounded.

<u>Pf</u>: (i) Let x, y both be limits of $(x_n)_{n \in \mathbb{N}}$.
~~will show $x = y$ by showing $d(x,y) < \varepsilon$ [illegible]~~
Let $\varepsilon > 0$. As limits $\exists N_x \in \mathbb{N}$ and $N_y \in \mathbb{N}$ s.t. $\forall n \geq N_x$
$$d(x_n, x) < \frac{\varepsilon}{2}$$
and $\forall n \geq N_y$
$$d(x_n, y) < \frac{\varepsilon}{2}$$

So by the triangle inequality: for $n \geq \max\{N_x, N_y\}$

$$d(x,y) \leq d(x, x_n) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have

$$0 \leq d(x,y) \leq 0 \Rightarrow d(x,y) = 0.$$

But then $x = y$.

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(ii) Let $(x_{n_k})_{k \in \mathbb{N}}$ be a subsequence, and let

$$x = \lim_{n \to \infty} x_n.$$

Let $\varepsilon > 0$, then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$d(x_n, x) < \varepsilon.$$

Since $n_1 < n_2 < \cdots$ is strictly increasing $\exists K \in \mathbb{N}$ s.t. $n_K \geq N$. Then $\forall k \geq K$ we have $n_k \geq n_K \geq N$. Thus

$$d(x_{n_k}, x) < \varepsilon$$

for all $k \geq K$. That is, $(x_{n_k})_{k \in \mathbb{N}}$ converges to x too.

(iii) Let $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$d(x_n, x) < 1.$$

Let $R = \max\{d(x_1, x), d(x_2, x), \ldots, d(x_{N-1}, x), 1\} \in \mathbb{R}$.

Claim: $\{x_n : n \in \mathbb{N}\} \subseteq B(x, R)$

Indeed, if $n \geq N$, then $d(x_n, x) < 1 \leq R$ so $x_n \in B(x, R)$. If $n < N$ then $n \in \{1, 2, \ldots, N-1\}$ and so $d(x_n, x) \leq R \Rightarrow x_n \in B(x, R)$. Thus the claimed inclusion holds and $\{x_n : n \in \mathbb{N}\}$ is bounded. $\square$

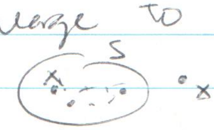
Remark: In a metric space (E, d), a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in E$ iff

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \quad x_n \in B(x, \varepsilon)$$

That is, convergence is defined in terms of open balls or more generally open sets. How does convergence relate to closed sets?

Thm Let S be a subset of a metric space (E,d). Then S is closed iff whenever $(x_n)_{n\in\mathbb{N}} \subseteq S$ is a convergent sequence with limit $x \in E$, we have $x \in S$.

· Remark: This theorem says sequences in closed sets S cannot converge to points outside of S:

Pic: 

Pf (Thm): ($\Longrightarrow$) Assume S is closed, so then S^c is open. Let $(x_n)_{n\in\mathbb{N}} \subseteq S$ be a sequence converging to some $x \in E$. Suppose, towards a contradiction, that $x \notin S$ i.e. $x \in S^c$. Since S^c is open, $\exists r > 0$ s.t. $B(x,r) \subseteq S^c$. Since $(x_n)_{n\in\mathbb{N}}$ conv to x, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$d(x_n, x) < r \iff x_n \in B(x,r)$$

But this contradicts $x_n \in S$. Thus we must have $x \in S$.

($\Longleftarrow$) ~~Assume all convergent sequences in S have their limits in S~~

We proceed by contrapositive. Suppose S is not closed. Consequently, S^c is not open, which means $\exists x \in S^c$ s.t. $\forall r > 0$ $B(x,r) \not\subseteq S^c$. This in turn means

$$B(x,r) \cap S \neq \emptyset \quad \forall r > 0.$$

We define a sequence in S as follows:

For each $n \in \mathbb{N}$, let x_n be any point from:

$$B(x, \tfrac{1}{n}) \cap S.$$

Thus $d(x, x_n) < \frac{1}{n}$ $\forall n \in \mathbb{N}$. We claim that $(x_n)_{n\in\mathbb{N}}$ ~~[illegible]~~ converges to x. Indeed, let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be s.t. $\frac{1}{N} < \varepsilon$. Then for all $n \geq N$ we have

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

and so $\forall n \geq N$

$$d(x, x_n) < \frac{1}{n} < \varepsilon.$$

Thus $(x_n)_{n\in\mathbb{N}} \subset S$ converges to $x \notin S$. $\square$

Sequences in $\mathbb{R}$

In the specific metric space $(\mathbb{R}, d)$ where

$$d(x,y) = |x-y| \qquad x,y \in \mathbb{R}$$

we have additional structure: operations $+, -, \times, \div$ and a natural ordering of elements. We will now explore how this additional structure interacts with the metric space structure (in particular, how it interacts with limits).

Prop: Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be sequences converging to $a, b \in \mathbb{R}$, respectively. Then:

(i) $\lim_{n\to\infty} (a_n + b_n) = a + b$

(ii) $\lim_{n\to\infty} (a_n - b_n) = a - b$

(iii) $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$

(iv) If $b \neq 0$ and $b_n \neq 0$ $\forall n \in \mathbb{N}$, then:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$$

<u>Proof</u>: (i) Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$\forall n \geq N_1 \quad |a_n - a| = d(a_n, a) < \frac{\varepsilon}{2}$$
$$\forall n \geq N_2 \quad |b_n - b| = d(b_n, b) < \frac{\varepsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$ we have both $n \geq N_1$ and $n \geq N_2$ so that

$$\begin{aligned} d(a_n+b_n, a+b) &= |a_n + b_n - (a+b)| \\ &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} (a_n + b_n) = a + b$.

9/22/2017 (ii) Note that by (iii) (which we will prove next), that

$$\lim_{n\to\infty} -b_n = \lim_{n\to\infty} (-1)\cdot b_n = \lim_{n\to\infty} (-1) \cdot \lim_{n\to\infty} b_n$$
$$= -1 \cdot b = -b$$

So appealing to (i) we have:

$$\begin{aligned} \lim_{n\to\infty} (a_n - b_n) &= \lim_{n\to\infty} (a_n + (-b_n)) \\ &\overset{(i)}{=} \lim_{n\to\infty} a_n + \lim_{n\to\infty} (-b_n) \\ &= a + -b = a - b. \end{aligned}$$

(iii) Let $\varepsilon > 0$.

Idea:
$$\begin{aligned} |a_n b_n - a \cdot b| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n(b_n - b)| + |(a_n - a)\cdot b| \\ &\leq \underbrace{|a_n|}_{\text{not too big}} \cdot \underbrace{|b_n - b|}_{\text{small}} + \underbrace{|a_n - a|}_{\text{small}} \cdot |b| \end{aligned}$$

Let $N_1 \in \mathbb{N}$ be s.t. $\forall n \geq N_1$

$$|a_n - a| < \frac{\varepsilon}{2|b|}$$

Now, $(a_n)_{n\in\mathbb{N}}$ is convergent and therefore bounded by previous Prop. So $\exists\, r>0$ and $x\in\mathbb{R}$ s.t.

$$a_n \in B(x,r) \quad \forall\, n\in\mathbb{N}$$

Moreover, we seen that this implies

$$x-r < a_n < x+r \quad \forall n\in\mathbb{N}$$

So:

$$a_n < x+r \le |x|+r$$
$$-a_n < -x+r \le |x|+r$$
$$\Longrightarrow \quad |a_n| \le |x|+r \quad \forall\, n\in\mathbb{N}.$$

So we now pick $N_2\in\mathbb{N}$ s.t. $\forall n\geq N$

$$|b_n-b| < \frac{\varepsilon}{2(|x|+r)}$$

Letting $N=\max\{N_1,N_2\}$, we have $\forall n\geq N$, using our previous estimate:

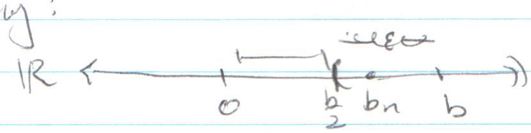
$$d(a_nb_n, ab) = |a_nb_n - ab|$$
$$\vdots$$
$$\le |a_n|\cdot|b_n-b| + |a_n-a|\cdot|b|$$
$$< (|x|+r)\cdot\frac{\varepsilon}{2(|x|+r)} + \frac{\varepsilon}{2|b|}\cdot|b|$$
$$= \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

So $\lim_{n\to\infty}(a_n b_n) = a\cdot b$.

(iv) We will show $\lim_{n\to\infty}\frac{1}{b_n} = \frac{1}{b}$, and the appeal to (iii). Let $\varepsilon>0$.

Idea: $\left|\frac{1}{b_n}-\frac{1}{b}\right| = \left|\frac{b-b_n}{b_n\cdot b}\right| = \frac{|b-b_n|}{|b|\cdot|b_n|}$ ← small ← stay away from zero?

We need to ensure that $|b_n|$ stays away from zero. We know this happens eventually:



Let $N_1\in\mathbb{N}$ be s.t. $\forall n\geq N$ $|b_n-b| < \frac{|b|}{2}$

By the reverse triangle inequality we have $\forall n \geq N$,

$$|b_n| = |b_n - b + b|$$
$$\geq \big|\, |b_n - b| - |b| \,\big|$$
$$= \big|\, |b| - |b_n - b| \,\big|$$

anticipate which term is larger

$$\geq |b| - |b_n - b|$$
$$> |b| - \frac{|b|}{2} = \frac{|b|}{2}.$$

So

$$* \quad |b_n| > \frac{|b|}{2} \qquad \forall n \geq N,$$

~~Now, let ε>0~~ so using our previous computation we have $\forall n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \cdots = \frac{|b - b_n|}{|b||b_n|} < \frac{|b - b_n|}{|b| \cdot \frac{|b|}{2}}$$
$$= \frac{2|b - b_n|}{|b|^2}$$

So we let $N_2 \in \mathbb{N}$ be s.t. $\forall n \geq N$

$$|b_n - b| < \frac{\varepsilon \cdot |b|^2}{2}$$

Then if $N = \max\{N_1, N_2\}$, $\forall n \geq N$ we have

$$d\left(\frac{1}{b_n}, \frac{1}{b}\right) = \left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{2|b_n - b_n|}{|b|^2} < \frac{2\,\frac{\varepsilon|b|^2}{2}}{|b|^2} = \varepsilon.$$

Thus $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Finally, using (iii) we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n}$$
$$= a \cdot \frac{1}{b} = \frac{a}{b}. \quad \square$$

Later (1) Ex: (1) $\left(\frac{5n+3}{n^2}\right)_{n\in\mathbb{N}} \to 0$ (2) $\frac{1}{n^2} \to 0$ (3) General tip: $\frac{4n^2}{3n^2+1}$

Next we examine how the ordering on $\mathbb{R}$ jives w/ limits:

<u>Prop:</u> Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be convergent sequences with limits $a, b \in \mathbb{R}$, respectively. If $a_n \leq b_n$ $\forall n \in \mathbb{N}$, then $a \leq b$.

Pf: By the previous prop:

$$b - a = \lim_{n\to\infty} b_n - a_n$$

Since $b_n - a_n \ge 0$ $\forall n$, we have

$$b_n - a_n \in [0,\infty) \quad \forall n \in \mathbb{N}.$$

Observe that $[0,\infty)$ is a closed set, Hence the limit of the sequence $(b_n - a_n)_{n\in\mathbb{N}}$ must be contained in this set: $b - a \in [0,\infty)$. That is, $b \ge a$. $\square$

Cor: If $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges to $c \in \mathbb{R}$ and ~~$a_n \le d$~~ $c \le a_n$ $\forall n \in \mathbb{N}$, then ~~$c \le d$~~ $c \le d$.

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~~Ex: The sequence~~

~~$$\left(\frac{5n+3}{n^2+1}\right)_{n\in\mathbb{N}}$$~~

~~converges to zero. Indeed, observe that~~

~~$$0 \le \frac{5n+3}{n^2+1} \le \frac{5n+3}{n^2}$$~~

~~By previous example: $\lim_{n\to\infty} \frac{5n+3}{n^2} = 0$.~~

~~Also,~~

Ex. (1) $\frac{1}{n^2}$ – two proofs: previous prop & direct. (2) $\frac{5n+3}{n^2}$ – previous prop (3) $\frac{4n}{3n^2+1}$ – general principle $\frac{1}{a+b} \le \frac{1}{a}, \frac{1}{b}$ (4) $\frac{4n^2}{3n^2+1}$ (5) $\frac{5n^2}{n^2-n}$ (6) 4, 5 in (n^2, d)

Prop (Squeeze Theorem)

Suppose $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ are sequences satisfying

$$a_n \le b_n \le c_n \quad \forall n \in \mathbb{N}$$

~~If $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ both converge to some $x \in \mathbb{R}$, then $(b_n)_{n\in\mathbb{N}}$~~

If $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ both converge to $x \in \mathbb{R}$, then $(b_n)_{n\in\mathbb{N}}$ also converges with $\lim_{n\to\infty} b_n = x$.

Pf: Let $\varepsilon > 0$. Let $N_1, N_2 \in \mathbb{N}$ be s.t.

$$\forall n \ge N_1 \quad |a_n - x| < \frac{\varepsilon}{2}$$

$$\forall n \ge N_2 \quad |c_n - x| < \frac{\varepsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$b_n - x \le \underbrace{c_n - b_n}_{\text{pos}} + b_n - x = c_n - x \le |c_n - x| < \frac{\varepsilon}{2}$$

and

$$x - b_n \le x - b_n + \underbrace{b_n - a_n}_{\text{pos}} = x - a_n \le |a_n - x| < \frac{\varepsilon}{2}$$

So $|b_n - x| < \varepsilon$ $\forall n \geq N$. Thus $(b_n)_{n\in\mathbb{N}}$ converges to x. $\square$

Ex: The sequence $\left(\frac{5n+3}{n^2+1}\right)_{n\in\mathbb{N}}$ converges to zero. Indeed, note

$$0 \leq \frac{5n+3}{n^2+1} \leq \frac{5n+3}{n^2} \quad \forall n \in \mathbb{N}.$$

Setting $a_n = 0$, $b_n = \frac{5n+3}{n^2+1}$, $c_n = \frac{5n+3}{n^2}$ for $n \in \mathbb{N}$. we have

$$\lim_{n\to\infty} a_n = 0 \quad \text{(constant seq)}$$

$$\lim_{n\to\infty} c_n = 0 \quad \text{(by previous ex.)}$$

Thus $\lim_{n\to\infty} \frac{5n+3}{n^2+1} = 0$ by the squeeze theorem.

Def: A sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is:

- increasing if $x_1 \leq x_2 \leq x_3 \leq \cdots$
- decreasing if $x_1 \geq x_2 \geq x_3 \geq \cdots$
- monotonic if it is either increasing or decreasing.

We say $(x_n)_{n\in\mathbb{N}}$ is strictly increasing/decreasing/monotonic if the above inequalities are strict.

Warning: Some people use "increasing" to mean strictly increasing, and "non-decreasing" to mean increasing. (similar for "decreasing"). ~~We~~
~~[illegible]~~

Thm A bounded monotonic sequence in $\mathbb{R}$ is convergent.

Pf: Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be bounded and monotonic. First, we consider the case when $(x_n)_{n\in\mathbb{N}}$ is increasing. Since the set $S := \{x_n : n \in \mathbb{N}\}$ is bounded, It has a supremum

$$x := \sup(S).$$

We claim the sequence converges to x.

Let $\varepsilon > 0$, then $\exists\, x_N \in S$ s.t.

$$x - \varepsilon < x_N \leq x. \quad \text{and } ~~x - x_N < \varepsilon~~$$

$$\Rightarrow x - x_N \leq \varepsilon$$

~~We have that $s = x_N$ for some N.~~ Then $\forall n \geq N$

$$x_N \leq x_n$$

$$\Longleftrightarrow \quad -x_n \leq -x_N.$$

So

$$|x_n - x| = x - x_n \leq x - x_N < \varepsilon.$$

So $\lim_{n\to\infty} x_n = x$.

If $(x_n)_{n\in\mathbb{N}}$ is decreasing, we define

$$x := \inf(S).$$

Let $\varepsilon > 0$, again $\exists\, x_N \in S$ st.

$$x \leq x_N < x + \varepsilon$$

$$\Rightarrow x_N - x < \varepsilon.$$

~~Now $s = x_N$ for some N~~, and $\forall n \geq N$ we have

$$x_n \leq x_N$$

So $|x_n - x| = x_n - x \leq x_N - x < \varepsilon$.

So $(x_n)_{n\in\mathbb{N}}$ converges to x. $\square$

~~10/2/2017~~

<u>Cor:</u> For $a \in \mathbb{R}$, if $|a| < 1$ then $\lim_{n\to\infty} a^n = 0$.

<u>Pf:</u> Observe that

$$d(a^n, 0) = |a^n - 0| = |a^n| = |a|^n = \big||a|^n - 0\big| = d(|a|^n, 0)$$

So if we can show $\lim_{n\to\infty} |a|^n = 0$, then it follows that $\lim_{n\to\infty} a^n = 0$.

Now, $0 \leq |a| < 1 \Rightarrow 0 \leq |a|^n < 1 \quad \forall n \in \mathbb{N}$. In particular the sequence is bounded. Also, the sequence is decreasing:

$$|a|^n - |a|^{n+1} = |a|^n(1 - |a|) \geq 0$$

so $|q|^n \geq |q|^{n+1}$. The theorem implies the limit exists. Denote

$$y := \lim_{n\to\infty} |q|^n$$

Then

$$|q| \cdot y = \left(\lim_{n\to\infty} |q|\right)\cdot\left(\lim_{n\to\infty} |q|^n\right) = \lim_{n\to\infty} |q|^{n+1} = y.$$

So $|q| \cdot y = y$. If $y \neq 0$, then $\Rightarrow |q| = 1$, contradicting $|q| < 1$. So we must have $y = 0$. $\square$

Completeness III.4

As we have seen, a sequence $(x_n)_{n\in\mathbb{N}}$ converges when all the points "cluster" together eventually. We can study this independently from the notion of convergence.

<u>Def</u>: In a metric space (E, d), a sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t.

$$\forall n, m \geq N \quad d(x_n, x_m) < \varepsilon$$

<u>Prop</u>: Any convergent sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (E, d) is also a Cauchy sequence.

<u>Pf</u>: Let $\varepsilon > 0$ and let $x = \lim_{n\to\infty} x_n$. Let $N \in \mathbb{N}$ be s.t.

$$\forall n \geq N \quad d(x_n, x) < \frac{\varepsilon}{2}$$

Then by the triangle inequality:

$$\forall n, m \geq N \quad d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

<u>Ex</u>: Consider the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ but in the metric space $E := \mathbb{R}\setminus\{0\}$ with the metric $d(x,y) = |x-y|$.

<u>Claim</u>: $(\frac{1}{n})_{n\in\mathbb{N}}$ in (E, d) is a Cauchy sequence but is <u>not</u> convergent.

Indeed, we know the sequence <u>wants</u> to converge