so $|q|^n \geq |q|^{n+1}$. The theorem implies the limit exists. Denote

$$y := \lim_{n\to\infty} |q|^n$$

Then

$$|q| \cdot y = \left(\lim_{n\to\infty} |q|\right)\cdot\left(\lim_{n\to\infty} |q|^n\right) = \lim_{n\to\infty} |q|^{n+1} = y.$$

So $|q| \cdot y = y$. If $y \neq 0$, then $\Rightarrow |q| = 1$, contradicting $|q| < 1$. So we must have $y = 0$. $\square$

Completeness III.4

As we have seen, a sequence $(x_n)_{n\in\mathbb{N}}$ converges when all the points "cluster" together eventually. We can study this independently from the notion of convergence.

Def: In a metric space (E, d), a sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t.

$$\forall n, m \geq N \quad d(x_n, x_m) < \varepsilon$$

Prop: Any convergent sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (E, d) is also a Cauchy sequence.

Pf: Let $\varepsilon > 0$ and let $x = \lim_{n\to\infty} x_n$. Let $N \in \mathbb{N}$ be s.t.

$$\forall n \geq N \quad d(x_n, x) < \frac{\varepsilon}{2}$$

Then by the triangle inequality:

$$\forall n, m \geq N \; d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Ex: Consider the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ but in the metric space $E := \mathbb{R}\setminus\{0\}$ with the metric $d(x,y) = |x-y|$.

Claim: $(\frac{1}{n})_{n\in\mathbb{N}}$ in (E, d) is a Cauchy sequence but is not convergent.

Indeed, we know the sequence wants to converge to zero, but since $0 \notin E$, $(\frac{1}{n})_{n\in\mathbb{N}}$ will not converge. To see that it is Cauchy, let $\varepsilon > 0$ and let $N > \frac{2}{\varepsilon}$. Then $\forall n \geq N$ we have

$$\frac{1}{n} \leq \frac{1}{N} < \frac{\varepsilon}{2}.$$

So $\forall n, m \geq N$

$$d(\tfrac{1}{n}, \tfrac{1}{m}) = |\tfrac{1}{n} - \tfrac{1}{m}| \leq \tfrac{1}{n} + \tfrac{1}{m} < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

<u>Prop</u>: Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space (E, d). Then:

(i) $\{x_n : n \in \mathbb{N}\}$ is bounded,

(ii) Any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is also a Cauchy sequence

(iii) If there exists a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ then $(x_n)_{n\in\mathbb{N}}$ is also convergent with

$$\lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k}.$$

<u>Pf</u>: (i) Let $\varepsilon = 1$ and let $N \in \mathbb{N}$ be s.t. $\forall n, m \geq N$

$$d(x_n, x_m) < 1.$$

Set $R = \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}$

Then $\forall n \in \mathbb{N}$, either $n \geq N$ in which case

$$d(x_n, x_N) < 1 \leq R$$

or $n < N$ in which case

$$d(x_n, x_N) \leq R.$$

Thus $\{x_n : n \in \mathbb{N}\} \subseteq B[x_N, R]$.

(ii) Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$

$$d(x_n, x_m) < \varepsilon.$$

Since $(n_k)_{k\in\mathbb{N}}$ is strictly increasing, $\exists K \in \mathbb{N}$ s.t. $n_K \geq N$. Consequently, $\forall k, \ell \geq K$ we have $n_k, n_\ell \geq n_K \geq N$ and so

$$d(x_{n_k}, x_{n_\ell}) < \varepsilon.$$

Thus $(x_{n_k})_{k\in\mathbb{N}}$ is Cauchy.

(iii) Suppose $x = \lim_{k\to\infty} x_{n_k}$. Let $\varepsilon > 0$, and let

~~N~~ $N \in \mathbb{N}$ be s.t. $\forall n, m \geq N$ $d(x_n, x_m) < \frac{\varepsilon}{2}$.
Let $K \in \mathbb{N}$ be s.t. $\forall k \geq K$, $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.
Fix $k_0 \geq K$ ~~be~~ s.t. $n_{k_0} \geq N$. Then $\forall n \geq N$ we have

$$d(x_n, x) \leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $(x_n)_{n\in\mathbb{N}}$ converges to x. $\square$

10/4/2017

Def: A metric space (E, d) is complete if every Cauchy sequence converges.

Non-Ex: $E = \mathbb{R}\setminus\{0\}$ with metric $d(x,y) = |x-y|$ is not complete by our previous example.

Prop Let (E, d) be a complete metric space, and let $S \subset E$ be a closed subset. Then $(S, d|_{S\times S})$ is a complete metric space.

Pf: Let $(x_n)_{n\in\mathbb{N}} \subseteq S$ be a Cauchy sequence with respect to $d|_{S\times S}$. But then $(x_n)_{n\in\mathbb{N}} \subset E$ and is also a Cauchy sequence w.r.t. d. Since E is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in E$. However, S is closed so must have $x \in S$ and so $(x_n)_{n\in\mathbb{N}}$ converges to x w.r.t. $d|_{S\times S}$. $\square$

Thm: $\mathbb{R}$ with the usual metric is complete.
Pf: Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a Cauchy sequence. Consider the set

$$S := \{x \in \mathbb{R} : x \leq x_n \text{ for any infinite number of } n \in \mathbb{N}\}$$

Now ~~Since~~ $(x_n)_{n\in\mathbb{N}}$ is Cauchy $\Rightarrow$ bounded. ~~In particular,~~ So S is non-empty since, in particular, any lower bound for $\{x_n : n \in \mathbb{N}\}$ will be in S.

On the other hand, $\{x_n : n \in \mathbb{N}\}$ is also bounded from above, and consequently S is bounded from above by any upperbound for $\{x_n : n \in \mathbb{N}\}$.
Set

$$x := \sup(S).$$

We claim $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$, Set $N \in \mathbb{N}$ s.t. $\forall n, m \geq N$

$$|x_n - x_m| < \frac{\varepsilon}{2}.$$

Now, ~~...~~ $x - \frac{\varepsilon}{2} \in S$
since $\exists s \in S$ satisfying

$$x - \frac{\varepsilon}{2} < s \leq x$$

and consequently for infinitely many $n \in \mathbb{N}$ we have

$$x_n \geq s > x - \frac{\varepsilon}{2}$$

i.e. $x - \frac{\varepsilon}{2} \in S$. ~~This means~~ But, since $x = \sup(S)$, $x + \frac{\varepsilon}{2} \notin S$.
So only finitely many $n \in \mathbb{N}$ satisfy

$$x + \frac{\varepsilon}{2} \leq x_n$$

while infinitely many satisfy $x - \frac{\varepsilon}{2} \leq x_n$.
Consequently, we can find $m \geq N$ s.t.

$$x - \frac{\varepsilon}{2} \leq x_m < x + \frac{\varepsilon}{2}$$

$$\iff \quad -\frac{\varepsilon}{2} \leq x_m - x < \frac{\varepsilon}{2}$$

$$\iff \quad |x_m - x| \leq \frac{\varepsilon}{2}$$

Thus $\forall n \geq N$ we have

$$|x_n - x| \leq |x_n - x_m| + |x_m - x|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{n\to\infty} x_n = x$, and hence $\mathbb{R}$ is complete. $\square$

<u>Cor:</u> For $n \in \mathbb{N}$, $\mathbb{R}^n$ with the n-dim'l Euc. metric d is complete.
<u>Pf:</u> Let $(\vec{x}_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ be a Cauchy sequence.
For each $k \in \mathbb{N}$, write

$$\vec{x}_k = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix} \in \mathbb{R}^n$$

Then for each coordinate $j = 1, \ldots, n$ we have

$$|x_j^{(k)} - x_j^{(l)}| \le \sqrt{(x_j^{(k)} - x_j^{(l)})^2}$$
$$\le \sqrt{(x_1^{(k)} - x_1^{(l)})^2 + \cdots + (x_n^{(k)} - x_n^{(l)})^2}$$
$$= d(\vec{x}_k, \vec{x}_l)$$

Thus for $\varepsilon > 0$, if we let $N \in \mathbb{N}$ be s.t. $\forall k, l \ge N$
$$d(\vec{x}_k, \vec{x}_l) < \varepsilon$$
then $\forall k, l \ge N$
$$|x_j^{(k)} - x_j^{(l)}| < \varepsilon.$$
That is, $(x_j^{(k)})_{k \in \mathbb{N}} \subseteq \mathbb{R}$ is a Cauchy sequence.
Since $\mathbb{R}$ is complete, it converges to some $x_j \in \mathbb{R}$.
We claim $(\vec{x}_k)_{k \in \mathbb{N}}$ converges to
$$\vec{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$
Indeed, let $\varepsilon > 0$. Let $N_1, N_2, \dots, N_n \in \mathbb{N}$ be s.t.
$$\forall k \ge N_j \quad |x_j^{(k)} - x_j| < \frac{\varepsilon}{\sqrt{n}}$$
Then, if $N = \max\{N_1, \dots, N_n\}$, $\forall k \ge N$ we have
$$d(\vec{x}_k, \vec{x}) = \sqrt{(x_1^{(k)} - x_1)^2 + \cdots + (x_n^{(k)} - x_n)^2} < \sqrt{\left(\frac{\varepsilon}{\sqrt{n}}\right)^2 + \cdots + \left(\frac{\varepsilon}{\sqrt{n}}\right)^2}$$
$$= \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \sqrt{\varepsilon^2} = \varepsilon.$$

So $\lim_{k \to \infty} \vec{x}_k = \vec{x}$ and $\mathbb{R}^n$ is complete. $\square$

Cor: $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are complete. Pf: the equiv

10/6/2017

Compactness III.5

Def In a metric space (E, d) for $S \subset E$, a collection $\{U_i\}_{i \in I}$ of open subsets $U_i \subseteq E$, $i \in I$, is called an open cover of S if
$$S \subset \bigcup_{i \in I} U_i$$
An open cover of E is simply called an open cover.
If $J \subseteq I$ and
$$S \subset \bigcup_{i \in J} U_i$$
then $\{U_i\}_{i \in J}$ is called a subcover of $\{U_i\}_{i \in I}$ of S