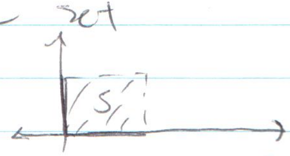
But this contradicts S_m not being covered by a finite union of the $U_i, i \in I$. $\square$

10/16/2017

Connectedness III.6

Before discussing "connectedness" we discuss the notion of relative open and closedness.

Ex: $(\mathbb{R}^2, d)$ be the 2-dim'l Euc. metric space. Consider the set

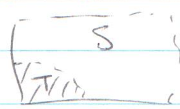


$S = \{(x,y) : 0 \le x < 1,\ 0 \le y < 1\}$

One can check that S is neither open nor closed in $(\mathbb{R}^2, d)$. However, in $(S, d|_{S\times S})$ S is automatically open and closed. Consider $B((x,y), r, d)$ in $(S, d|_{S\times S})$

single example

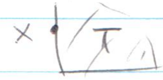
If we look at:



$T = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1/4,\ x \ge 0,\ y \ge 0\} \subseteq S$

Then T is neither open or closed in $(\mathbb{R}^2, d)$ but in $(S, d|_{S\times S})$ it is open.

For



The open balls in $\underline{\underline{S}}$ centered at x cannot leak out to the left because when we restrict to S there is nothing to the left.

The above example leads to the following definition

Def: In a metric space (E, d), for $T \subset S \subset E$ we say T is open (resp. closed) relative to S

if T is open (resp. closed) in the metric space $(S, d|_{S\times S})$.

Ex: $\{0\} \subseteq \mathbb{R}$ is not open, but relative to $\mathbb{Z}$ it is. Indeed, set $r = \frac{1}{2}$. Then in $(\mathbb{Z}, d)$

$$B(0, \tfrac{1}{2}) = \{x \in \mathbb{Z} : |x - 0| < \tfrac{1}{2}\} = \{0\}.$$

Def: In a metric space (E, d), a subset $S \subseteq E$ is connected if the only subsets of ~~this~~ that are open and closed relative to S are $\emptyset$ and S. We say S is disconnected if it is not connected.

Remark: ~~If S is disconnected, $\exists A \subseteq S$ which is open and closed rel. to S. Hence $B = S \setminus A$ is open and closed relative to S. That is, $S = A \cup B$, $A \cap B$, and A, B are both open relative to S. This is in fact equivalent~~

Prop: S is disconnected iff $\exists$ non-empty $A, B \subseteq S$ subsets open relative to S s.t.

$$A \cap B = \emptyset \quad \text{and} \quad A \cup B = S.$$

Pf: $(\Rightarrow)$ If S is disconnected, $\exists A \subseteq S$ open and closed relative to S and s.t. $\emptyset \neq A \neq S$. Thus $B := S \setminus A$ is nonempty and open relative to S (since A is closed). By def we have $A \cap B = \emptyset$ and $A \cup B = S$.

$(\Leftarrow)$ ~~A~~ $B \neq \emptyset \Rightarrow A \neq S$. Since B is open rel. to S, A is closed rel. to S. Thus ~~A is open~~ $\emptyset \subsetneq A \subsetneq S$ is open & closed rel. to S $\square$

Prop: In $(\mathbb{R}, d)$ with usual metric, if $S \subseteq \mathbb{R}$ contains $a<b$ but $\exists\, c \notin S$ such that $a<c<b$, then S is disconnected.

Pf: First observe $c \notin S$ implies

$$S \subseteq \mathbb{R} \setminus \{c\} = (-\infty, c) \cup (c, +\infty).$$

But then

$$S = (S \cap (-\infty, c)) \cup (S \cap (c, +\infty)).$$

Since $(-\infty, c)$ and $(c, +\infty)$ are open in $\mathbb{R}$, $S \cap (-\infty, c)$ and $S \cap (c, +\infty)$ are open relative to S by Homework #7. They are also non-empty since they contain a and b, respectively. By the previous prop., we see that S is disconnected. $\square$

Theorem In $\mathbb{R}$ with the usual metric, any open, closed, or half-open interval (including $\mathbb{R} = (-\infty, \infty)$) is connected.

Pf: We handle all cases simultaneously by showing $S \subseteq \mathbb{R}$ is connected so long as whenever $a, b \in S$ satisfy $a<b$, then $[a,b] \subseteq S$. (Note that all intervals satisfy this.)

Suppose $S \subseteq \mathbb{R}$ satisfies this property, but further ~~assume~~ suppose, towards a contradiction, that S is disconnected. Hence we can find non-empty, disjoint, subsets $A, B \subseteq S$ that are open relative to S and satisfy

$$S = A \cup B.$$

Pick $a \in A$, $b \in S$. wlog, $a<b$. Then By assumption $[a,b] \subseteq S$.

Set

$$A_1 = A \cap [a,b]$$
$$B_1 = B \cap [a,b]$$

Then A_1 and B_1 are ~~non~~-empty, disjoint, relatively open subsets of $[a,b]$ satisfying $[a,b] = A_1 \cup B_1$. Since B_1 is open relative to $[a,b]$, A_1 is closed relative to $[a,b]$. By the Homework, $\exists\, V \subseteq \mathbb{R}$ closed s.t.

$$A_1 = [a,b] \cap V \Rightarrow A_1 \text{ is closed}$$

in $\mathbb{R}$. Since A_1 is bounded above (by b), ~~sup~~ $c := \sup(A_1)$ exists and is in A_1. ~~Since~~ ~~$b \notin A_1$~~ Then $c \le b$, but $b \notin A_1$, so $c < b$. Now, on the other hand A_1 is open rel. to $[a,b]$ so by Homework again:

$$A_1 = [a,b] \cap U$$

for some $U \subseteq \mathbb{R}$ open. This means for some $0 < \varepsilon < b-c$, we must have

$$(c-\varepsilon, c+\varepsilon) \subseteq A_1$$

but this contradicts $c = \sup(A_1)$. $\square$

Once we discuss continuous functions, and the relation to connected sets, we will be able to produce many more examples of connected sets:

10/18/2017

Prop: Let (E,d) be a metric space. ~~Suppose~~ Let $\{S_i\}_{i \in I}$ be a collection of connected subsets of E. Suppose $\exists\, i_0 \in I$ s.t.

$$S_{i_0} \cap S_i \neq \emptyset \quad \forall i \in I$$

Then $\bigcup_{i \in I} S_i$ is connected.

Pf: Similar to HW, left as exercise. $\square$

~~Ex~~ Ex: