Prop (Chain Rule)

Let $U, V \subseteq \mathbb{R}$ be open. Let $f: U \to V$, $g: V \to \mathbb{R}$. For $x_0 \in U$, assume f diff'ble at x_0, and that g is diff'ble at $f(x_0)$. Then $g \circ f$ is diff'ble at x_0 with

$$(g\circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Pf: Define $Q: V \to \mathbb{R}$ by

$$Q(y) = \begin{cases} \dfrac{g(y) - g(f(x_0))}{y - f(x_0)} & \text{if } y \in V \setminus \{f(x_0)\} \\ g'(f(x_0)) & \text{if } y = f(x_0) \end{cases}$$

By def. of $g'(f(x_0))$, Q is cts at $\cancel{g}\, f(x_0)$. Since the comp. of cts functions is cts, we have that $Q \circ f$ is cts at x_0.

Thus

$$(g\circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim_{x \to x_0} \frac{Q(f(x))(f(x) - f(x_0))}{x - x_0}$$

$$= Q(f(x_0)) \cdot f'(x_0)$$

$$= g'(f(x_0)) \cdot f'(x_0). \quad \square$$

The Mean Value Theorem V.3

Prop: Let $f: U \to \mathbb{R}$ with $U \subseteq \mathbb{R}$ open. Assume f attains a minimum or maximum value at $x_0 \in U$. If f is diff'ble at x_0, then $f'(x_0) = 0$.

Pf: Suppose, TAC, $f'(x_0) \neq 0$. Then letting $\varepsilon = \frac{|f'(x_0)|}{2}$, $\exists \delta > 0$ s.t. if $x \neq x_0$ satisfies $|x - x_0| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \frac{|f'(x_0)|}{2}$$

$$\Longleftrightarrow \quad f'(x_0) - \frac{|f'(x_0)|}{2} < \frac{f(x) - f(x_0)}{x - x_0} < f'(x_0) + \frac{|f'(x_0)|}{2}$$

$$\left.\begin{matrix} \frac{f'(x_0)}{2} & \text{if } f'(x_0) > 0 \\ \frac{3f'(x_0)}{2} & \text{if } f'(x_0) < 0 \end{matrix}\right\} = \hookleftarrow \qquad \hookrightarrow \begin{cases} \frac{3f'(x_0)}{2} & \text{if } f'(x_0) > 0 \\ \frac{f'(x_0)}{2} & \text{if } f'(x_0) < 0 \end{cases}$$

In either case, both endpoints have the same sign, which means $\frac{f(x)-f(x_0)}{x-x_0}$

never changes sign for $|x-x_0|<\delta$.

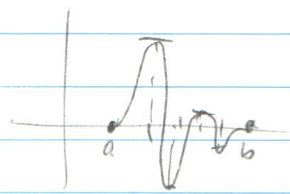
However, if $f(x_0)$ is maximum, $f(x)-f(x_0)\le 0$ always, while $x-x_0$ can be either positive or neg., a contradiction. Similarly if $f(x_0)$ is a min.

Thus we must have $f'(x_0)=0$ $\square$

<u>Lemma (Rolle's Thm)</u>

Let $a,b\in\mathbb{R}$, $a<b$ and let $f:[a,b]\to\mathbb{R}$ be cts on $[a,b]$, diff'ble on (a,b).

If $f(a)=f(b)=0$, then $\exists\, c\in(a,b)$ s.t. $f'(c)=0$.



<u>Pf</u> Since $[a,b]$ is compact, f attains a minimum and maximum on $[a,b]$, say max at c_1 and min at c_2.

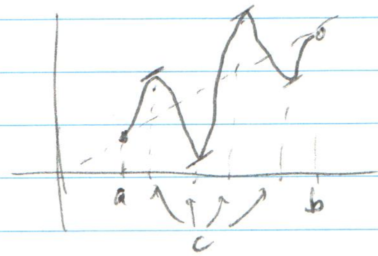
If either $c_1\in(a,b)$ or $c_2\in(a,b)$ then $f'(c_i)=0$ by prop. Otherwise, $c_1,c_2\in\{a,b\}$ which means

$$\left.\begin{aligned}\max_{x\in[a,b]} f(x)&=f(c_1)=0\\ \min_{x\in[a,b]} f(x)&=f(c_2)=0\end{aligned}\right\}\Rightarrow f(x)=0\quad\forall x\in[a,b]$$

Consequently $f'(x)=0$ $\forall x\in[a,b]$. $\square$

<u>Thm (Mean Value Theorem)</u>

Let $a,b\in\mathbb{R}$, $a<b$, and let $f:[a,b]\to\mathbb{R}$ be cts on $[a,b]$ and diff'ble on (a,b). Then $\exists\, c\in(a,b)$ s.t. $\frac{f(b)-f(a)}{b-a}=f'(c)$.



Pf: Define

$$g(x) = \left(\frac{f(b)-f(a)}{b-a}\right)\cdot(x-a) + f(a),$$

which is a linear function.
Consider $h(x) = f(x) - g(x)$.
Then h is cts on $[a,b]$, diff'ble on (a,b) and

$$h(a) = f(a) - g(a) = f(a) - f(a) = 0$$

$$h(b) = f(b) - g(b) = f(b) - \left(\frac{f(b)-f(a)}{b-a}\right)(b-a) - f(a)$$
$$= 0.$$

By Rolle's thm, $\exists\, c \in (a,b)$ s.t.

$$0 = h'(c) = f'(x) - g'(c)$$
$$= f'(c) - \frac{f(b)-f(a)}{b-a}$$

That is,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \square$$

10 The MVT allows us to transfer information from f' to f:

Cor 1: If $f:(a,b) \to \mathbb{R}$ is diff'ble and $f'(x) = 0$ $\forall x \in (a,b)$, then f is a constant function.

Pf: For any $x_1, x_2 \in (a,b)$ say $x_1 < x_2$
Then $\exists\, c \in (x_1, x_2)$ s.t.

$$0 = f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

$$\iff 0 = f(x_1) - f(x_2)$$
$$\iff f(x_1) = f(x_2).$$

Since $x_1, x_2 \in (a,b)$ were arbitrary we have that f is constant. $\square$

Cor 2: If $f, g : (a,b) \to \mathbb{R}$ are diff'ble and $f'(x) = g'(x)$ $\forall x \in (a,b)$, then $f(x) = g(x) + c$ for some constant $c \in \mathbb{R}$.

Pf: $(f-g)'(x) = 0$ $\forall x \in (a,b) \Rightarrow f - g = c$. $\square$

Cor 3: If $f : (a,b) \to \mathbb{R}$ is diff'ble and

if $f'(x)$ is $\begin{cases} \text{strictly pos.} \\ \text{positive} \\ \text{strictly neg} \\ \text{negative} \end{cases}$ $\forall x \in (a,b)$ then f is $\begin{cases} \text{strictly inc.} \\ \text{increasing} \\ \text{strictly dec.} \\ \text{decreasing} \end{cases}$

Pf: For $a < x_1 < x_2 < b$, the sign of

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is determined by the sign of $f(x_2) - f(x_1)$, but the above $= f'(c)$ for some $c \in (x_1, x_2)$.

So e.g. if $f'(c) > 0 \Rightarrow f(x_2) - f(x_1) > 0$. $\square$

Remark: The converse is not true in general: $f(x) = x^3$ is strictly increasing but $f'(0) = 0$.

Taylor's Theorem V.4

Higher order derivatives: Suppose $f : U \to \mathbb{R}$ is diff'ble and that $f' : U \to \mathbb{R}$ is also diff'ble. We say f is twice differentiable and write

$$(f')' = f'' = \frac{d^2}{dx^2}(f) = \frac{d^2 f}{dx^2}$$

If $f'' : U \to \mathbb{R}$ is diff'ble, we say f is three times diff'ble and write

$$(f'')' = f^{(3)} = \frac{d^3}{dx^3}(f) = \frac{d^3 f}{dx^3}$$

In general, for $n \in \mathbb{N}$ we say f is n times