$$\sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}) = \sum_{i=1}^{N} x_i - x_{i-1} = b-a.$$

(ii) Since $\mathbb{R}\setminus\mathbb{Q}$ is dense, can also pick
$x_i' \in (\mathbb{R}\setminus\mathbb{Q}) \cap [x_{i-1}, x_i]$ $i = 1,\dots,N$
Then,
$$\sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}) = \sum_{i=1}^{N} 0 = 0.$$
Thus there is no $A \in \mathbb{R}$ s.t. $|S - A|$ small.

~~Prop: If f is Riemann integrable on [a,b], then its integral is unique.~~

Remark: The ~~role of~~ 'x' in $\int_a^b f(x)\,dx$ is meaningless. We can replace it by any character we want:
$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \int_a^b f(☺)\,d☺$$

11/29/2017

Linearity and order Properties of the Integral VI.2

Prop: (i) If $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ then so is $f+g$ with
$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(ii) If $f : [a,b] \to \mathbb{R}$ is int'ble on $[a,b]$ ~~and $c \in \mathbb{R}$~~ then so is $c \cdot f$ for any $c \in \mathbb{R}$ with
$$\int_a^b (c \cdot f)(x)\,dx = c \int_a^b f(x)\,dx.$$

Pf: (i) Let $\varepsilon > 0$. Then f, g int'ble means $\exists\ \delta_1, \delta_2 > 0$ st. if S_1 and S_2 are Riemann sums of f and g corresponding to partitions of width

$< \delta_1$ and $< \delta_2$, respectively, then

$$\left|\int_a^b f(x)\,dx - S_1\right| < \frac{\varepsilon}{2} \qquad \left|\int_a^b g(x)\,dx - S_2\right| < \frac{\varepsilon}{2}$$

set $\delta = \min\{\delta_1, \delta_2\}$ and let $a = x_0 < x_1 < \cdots < x_N = b$ be a partition of width $< \delta$. select rep's $x_i^* \in [x_{i-1}, x_i]$ $i = 1, \ldots, N$. Then

$$\left|\sum_{i=1}^{N} (f+g)(x_i^*)(x_i - x_{i-1}) - \left(\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right)\right|$$

$$\leq \left|\sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}) - \int_a^b f(x)\,dx\right| + \left|\sum_{i=1}^{N} g(x_i^*)(x_i - x_{i-1}) - \int_a^b g(x)\,dx\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f+g$ is int'ble w/ claimed integral

(ii) If $c = 0$, this is immediate by example 1. Otherwise, given $\varepsilon > 0$ choose δ as in def'n of integrability for $\frac{\varepsilon}{|c|}$. $\square$

<u>Note</u>: It follows from (i) and (ii) above (for $c = -1$) that

$$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

<u>Prop</u>: If $f : [a,b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$.

<u>Pf</u>: Let $\varepsilon > 0$. Since f is integrable, $\exists$ a Riemann sum

$$S = \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1})$$

s.t.

$$\left| S - \int_a^b f(x)\,dx \right| < \varepsilon$$

Observe that since $f(x) \geq 0$, $S \geq 0$.

Thus

$$\int_a^b f(x)\,dx \geq S - \varepsilon \geq -\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we must have
$$\int_a^b f(x)\,dx \geq 0. \qquad \square$$

<u>Cor</u> If $f, g : [a,b] \to \mathbb{R}$ are integrable and $f(x) \leq g(x)$ $\forall x \in [a,b]$, then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

<u>Pf</u>: Apply the previous prop to $h = g - f$:
$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b (g-f)(x)\,dx \geq 0. \qquad \square$$

<u>Cor</u> If $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and $\exists\, m, M \in \mathbb{R}$ s.t.
$$m \leq f(x) \leq M \qquad \forall x \in [a,b]$$
then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$
<u>Pf</u>: Apply the previous cor and Ex 1:
$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a) \qquad \square$$

Existence of the Integral VI.3

(Cauchy-criterion for Riemann sums)

<u>Lemma 1</u>: $f : [a,b] \to \mathbb{R}$ is int'ble iff $\forall \varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|S_1 - S_2| < \varepsilon$ whenever S_1 and S_2 are Riemann sums for f corresponding to partitions of $[a,b]$ of width less than δ.

<u>Pf</u>: ($\Rightarrow$) Suppose f is int'ble. Let $\varepsilon > 0$, and let $\delta > 0$ be s.t.
$$\left| S - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{2}$$
whenever S is a Riemann sum for f corresponding to a partition of width $< \delta$.