12.6.2017. Review Part 1.



Basic Objects:

① Metric Space

$(E, d)$ where $E$ a set and $d: E \times E \to \mathbb{R}$ s.t

d represents a distance within E.

i) $d(x,y) \geq 0$. $\forall x, y \in E$.

ii) $d(x,y) = 0$ iff $x = y$.

iii) $d(x,y) = d(y,x)$ $\forall x, y \in E$.

iv) $d(x,y) \leq d(x,z) + d(z,y)$. $\forall x, y, z \in E$.

② Continuous functions.

Two metric spaces $(E, d)$ and $(E', d')$.

$f: E \to E'$.

- $f$ is <u>continuous at $x_0 \in E$</u> if $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t if $d(x, x_0) < \delta$, for $x \in E$ then $d'(f(x), f(x_0)) < \varepsilon$.
- $f$ is <u>continuous on $E$</u> if it's cts at every $x_0 \in E$.
- $f$ is <u>uniformly continuous on $E$</u> if $\forall \varepsilon > 0$. $\exists \delta > 0$ s.t if $d(x,y) < \delta$ for any $x, y \in E$. then $d'(f(x), f(y)) < \varepsilon$.

difference is that in regular cty. $\delta$ can depend on $x_0$. in unif cty, $\delta$ works for all $x, y \in E$.

③ Limits of Sequences

$(E, d)$ is a metric space

a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$. converges to $x_0 \in E$ if $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$ s.t $d(x, x_0) < \varepsilon$.

philosophy for choosing $N$: balance simplicity and efficacy.

Convergent sequences: · have unique limit.

· are <u>bounded</u>.

($\exists y \in E$ and $R > 0$ s.t $x_n \in B(y, R)$ $\forall n \in \mathbb{N}$).

· all subsequences are convergent and have the same limit.

$\star$ $f: E \to E'$ is cts at $x_0 \Leftrightarrow \forall (x_n)_{n \in \mathbb{N}} \subseteq E$ converging to $x_0 \in E$. $(f(x_n))_{n \in \mathbb{N}} \subseteq E'$ converges to $f(x_0)$

④ Open and Closed Balls.

For $(E, d)$ a metric space, $x \in E$ and $r > 0$.

open ball. $\longrightarrow$ $B(x, r) = \{y \in E,\ d(x,y) < r\}$. $\ni x$

closed ball $\longrightarrow$ $B[x, r] = \{y \in E,\ d(x,y) \leq r\}$. $\ni x$.

$f: E \to E'$ is cts at $x_0 \in E$ $\iff$ $\forall$ open ball $B(f(x_0), \varepsilon) \subseteq E'$ centered at $f(x_0)$, $\exists$ an open ball $B(x_0, \delta) \subseteq E$ s.t $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$

$(x_n)_{n \in \mathbb{N}} \subseteq E$ converges to $x \in E$ $\iff$ For every open ball $B(x, \varepsilon)$ centered at $x$, only finitely many indices $n \in \mathbb{N}$ satisfy $x_n \notin B(x, \varepsilon)$



⑤ Open Sets:

$S \subseteq E$ is <u>open</u> if $\forall x \in S$, $\exists r > 0$ s.t $B(x, r) \subseteq S$.

* every point in S has room to wiggle around without leaving S.

Exs of open sets:

- $B(x, r)$ open balls.
- $\emptyset$. $E$. always open.
- $(a, b) \subseteq \mathbb{R}$ open.

- Arbitrary unions of open sets are open.
- <u>Finite</u> intersections of open sets are open.

The previous result breaks for infinite intersections.

Ex: $S_n := (-\frac{1}{n}, \frac{1}{n})$ $n \in \mathbb{N}$.
↳ open.

But $\bigcap_{n \in \mathbb{N}} S_n = \{0\}$ ← not open.

* Openness corresponds strict inequality.
which are not preserved under limits.

$f: E \to E'$. $f$ cts on $E$ $\iff$ $\forall U \subseteq E'$ open, $f^{-1}(U) = \{x \in E, f(x) \in U\} \subseteq E$ is open.

This characterization / definition is most useful when working with abstract metric spaces & functions.

⑥ Closed sets.

$S \subseteq E$ is <u>closed</u> if either / both of the following hold(s).

i) $E \setminus S = S^c$ (the complement of $S$) is open.

ii) $\forall$ convergent sequences $(x_n)_{n \in \mathbb{N}} \subseteq S$, you also have $\lim_{n \to \infty} x_n \in S$.

* which definition you use depends whether you "know more" about $S$ or $S^c$

Non - Ex: $S = [0,1)$

$(1-\frac{1}{n})_{n\in\mathbb{N}} \subseteq S$, but $\lim_{n\to\infty}(1-\frac{1}{n}) = 1 \notin S$.

So $S$ is not closed.

Exs of closed sets:

- $B[x,r]$ closed balls
- $\phi$, $E$. always closed.
- $[a,b] \subseteq \mathbb{R}$ closed.

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Ex: $S_n = [\frac{1}{n}, 1-\frac{1}{n}]$. $n \geq 2$.

$\bigcup_{n=2}^{\infty} S_n = (0,1)$

So only finite unions in previous result.

$f: E \to E'$ is cts on $E \iff \forall V \subseteq E'$ closed, $f^{-1}(V) = \{x \in E, f(x) \in V\} \subseteq E$ is closed.

Warning: Sets can be
a) open but not closed. $(0,1)$.
b). closed but not open. $[0,1]$.
c) closed and open (i.e. clopen) $\phi$, $\mathbb{R}$.
d) neither open nor closed. $[0,1)$

e.g. not open $\not\Rightarrow$ closed.

## 7. Cauchy Sequences.

$(x_n)_{n\in\mathbb{N}} \subseteq E$ is cauchy if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall n, m \geq N$, $d(x_n, x_m) < \varepsilon$.

* Our sequences is clustering up, but we might not know which point (i.e. the limit) they clustering around.

Convergent sequences are always cauchy, reverse not true (convergent sequence $\Rightarrow$ cauchy sequence, $\not\Leftarrow$)

Ex: $E = [0,1)$.

then $(1-\frac{1}{n})_{n\in\mathbb{N}}$ is cauchy but not convergent in $(E,d)$

* $f: E \to E'$ is unif. cts $\Rightarrow$ $(x_n)_{n\in\mathbb{N}} \subseteq E$ cauchy, then $(f(x_n))_{n\in\mathbb{N}} \subseteq E'$ cauchy.

## ⑧ Complete

A metric space $(E, d)$ is complete if every cauchy sequence actually does converge.

Ex: $\mathbb{R}$. $\mathbb{R}^n$.

Non-Ex: $\mathbb{Q}$. $[0,1)$

- If $E$ is complete, and $S \subseteq E$ is closed, then $S$ is complete.

## Compact :

$S \subseteq E$ is compact if any / all of the following holds.

i) Every open cover for $S$ has a finite subcover.

ii) Every sequence $(x_n)_{n \in \omega} \subseteq S$ has a convergent subsequence.

iii) $S$ is complete and totally bounded.

$\forall \varepsilon > 0$, $S$ can be covered by finitely many closed balls of radius $\varepsilon$.

· compact sets are always closed. complete. and bounded.

Ex: (Heine-Borel) Every closed and bounded subset of $\mathbb{R}^n$ is compact.

· If $E$ is compact, then every infinite set has a cluster point

$x$ is a cluster point of $S$ if $\forall r > 0$, $B(x,r) \cap (S \setminus \{x\}) \neq \emptyset$

$\Leftrightarrow \forall r > 0$. $B(x,r) \cap S$ has infinitely many elements

· $f : E \to E'$ is cts. $S \subseteq E$ is compact. then $f(S) \subseteq E'$ is compact.

· If $S \subseteq E$ is compact, $f$ is automatically uniformly cts on $S$.

## Connected.

$T \subseteq S \subseteq E$. say $T$ is open relative to $S$ if :

$$\forall x \in T,\ \exists r > 0 \text{ s.t } B_S(x,r) := \{ y \in S : d(x,y) < r \} \subseteq T$$

$$= (B(x,r) \cap S)$$

$\Leftrightarrow$ $\exists U \subseteq E$ open s.t $T = U \cap S$.

We say $T$ is closed relative to $S$ if

$S \setminus T$ is open relative to $S$.

$\Leftrightarrow$ $\exists V \subseteq E$ closed s.t $T = V \cap S$.

$\Leftrightarrow$ $\forall (x_n)_{n \in \omega} \subseteq T$ converging to some $x \in S$, we have $x \in T$.

(There could be sequences in $T$ converging to points in $S^c$, but ~~there exists~~ these aren't relevant here)

Ex: $\phi, S$ are always both open and closed relative to $S$.

- Say $S \subseteq E$ is connected if $\phi$ and $S$ are the only subsets of $S$ that are both open and closed relative to $S$.

- Say $S \subseteq E$ is disconnected if it is not connected.

  $\Leftrightarrow$ $\exists\ \phi \subsetneq A \subsetneq S$ that is both open & closed rel. to $S$.

  $\Leftrightarrow$ $\exists$ non-empty, disjoint, rel. open subset $A, B \subseteq S$ s.t $A \cup B = S$.

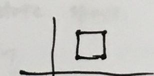
Ex: Intervals in $\mathbb{R}$ are connected.

- $f: E \to E'$ is cts and $S \subseteq E$ is connected, then $f(S)$ is connected in $E'$.

  Ex: Any cts image of an interval is connected.

- Unions of connected sets w/ non-empty intersections are connected.

  Ex: In $\mathbb{R}^2$.

 - connected.

Sequence of functions.

$(E, d)$ $(E', d')$ metric spaces.

for each $n \in \mathbb{N}$, $f_n : E \to E'$.

$f : E \to E'$.

- We say $(f_n)_{n \in \mathbb{N}}$ converges to $f$ at $x_0 \in E$ if

$$\lim_{n \to \infty} \overset{E'}{\downarrow}\ f_n(x_0) = f(x_0)$$

$\Leftrightarrow \forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ s.t $\forall n \geq N,\ d'(f_n(x_0), f(x_0)) < \varepsilon$.

- We say $(f_n)_{n \in \mathbb{N}}$ converges to $f$ pointwise on $E$. if

$\forall x \in E,\ \lim_{n \to \infty} f_n(x) = f(x)$.

$\Leftrightarrow$ $\underline{\forall x \in E}$, $\forall \varepsilon > 0$. $\underline{\exists N \in \mathbb{N}}$ s.t $\forall n \geq N$, $d'(f_n(x), f(x)) < \varepsilon$.

ordering matters: $N$ potentially depends on $x$.

· We say $(f_n)_{n\in\mathbb{N}}$ converges to $f$ uniformly on $E$. if

$$\forall \varepsilon>0,\ \underline{\exists N\in\mathbb{N}}\ \text{s.t}\ \forall n\geq N,\ \underline{\forall x\in E},\ d'(f_n(x), f(x)) < \varepsilon.$$

N $\underline{\text{cannot}}$ depend on x.

same N should work for all $x\in E$.

$$\Leftrightarrow \forall \varepsilon>0,\ \exists N\in\mathbb{N}\ \text{s.t}\ \forall n\geq N\quad \sup\{d'(f_n(x), f(x)): x\in E\} < \varepsilon.$$

· If each $f_n$ is cts and $(f_n)_{n\in\mathbb{N}}$ convergent uniformly to $f$. then $f$ is cts.

· if $E'$ is complete.

$(f_n)_{n\in\mathbb{N}}$ convergent unif. iff (Cauchy criterian).

$$\forall \varepsilon>0.\ \exists N\in\mathbb{N}\ \text{s.t}\ \forall n,m\geq N,\ \sup\{d'(f_n(x), f(x)): x\in E\} < \varepsilon$$

· E - compact.

$C(E,E') = \{f: E\to E' : f \text{ is continuous}\}.$

$D(f,g) := \sup\{d'(f(x), g(x)) : x\in E\}.$

$(C(E,E'), D)$ metric space.

where ? conv. in this metric space $\Leftrightarrow$ unif. conv. of the functions.

Moreover, if $E'$ is complete, then $C(E,E')$ is complete.

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* If E is not compact. $f,g: E\to E'$.

$\sup\{d'(f(x), g(x)): x\in E\}$ might not exist.

Ex: $E=\mathbb{R}$. $f,g:\mathbb{R}\to\mathbb{R}$.

$f(x)=1.$ $g(x)=x$

$\{|1-x| : x\in\mathbb{R}\}$ not bounded. so no sup.

$\rho(1,x)=\infty.$

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The Real Numbers.

· Properties of $\mathbb{R}$ : · field prop. $(+, -, \times, \div, 1, 0)$.

· order prop : $(a\leq b,\ a<b)$.

· Least Upper Bound property.

Every non-empty subset $S\subseteq\mathbb{R}$ that is bounded from above has a least upper bound / supremum. $\sup(S)$ exists.

$\Leftrightarrow$ Greatest lower bound property

Every non-empty subset $S\subseteq\mathbb{R}$ that is bounded from below has a greatest lower bound / infimum.

inf / sup: For $S$ non-empty. bounded.

$\forall \varepsilon > 0.\ \exists x\ y \in S.$ s.t

$$\left.\begin{array}{l} \inf(S) \le x < \inf(S) + \varepsilon. \\ \\ \sup(S) - \varepsilon < y \le \sup(S) \end{array}\right\}$$

$S$ may not contain its $\inf(S)$ and $\sup(S)$. but it always has elt close to them.

- For $S \subseteq \mathbb{R}$ closed. always have $\inf(S).\ \sup(S) \in S$.

Sequence in $\mathbb{R}$.

can do limit arithmetic:

$$\lim_{n\to\infty} (a_n \cdot b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$$

provided both $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converge.

- If $(a_n)_{n\in\mathbb{N}}$ is bounded and monotonic

then it converges to either

$\inf \{a_n : n \in \mathbb{N}\}$.

or

$\sup \{a_n : n \in \mathbb{N}\}$.

depending on the type of monotonicity

Compactness. (Heine-Borel): closed. bounded subsets in $\mathbb{R}^n$ are compact.

Ex. $[a,b]$ - closed interval is compact.

- $f: [a,b] \to \mathbb{R}$ cts. $f$ attains its min & max:

$\exists\ x_1.\ x_2 \in [a,b]$ s.t

$$f(x_1) = \inf \{f(x) : x \in [a,b]\}.$$

$$f(x_2) = \sup \{f(x) : x \in [a,b]\}.$$

Recall. $f: E \to \mathbb{R}^n$.

$f$ cts iff $\pi_1 \circ f, \pi_2 \circ f, \dots \pi_n \circ f$ are cts.

Completeness: $\mathbb{R}^n$ complete $\forall n \in \mathbb{N}$.

Connected: Any interval in $\mathbb{R}$ is connected.

Intermediate Value Theorem:

If: $a, b \in \mathbb{R}$. $a < b$

$f: [a,b] \to \mathbb{R}$ cts. then for every $y$ between $f(a)$ and $f(b)$.

$\exists x \in (a,b)$ s.t $f(x) = y$.

(no gaps in range of cts functions defined on intervals)