Notation

5.1

- $\mathcal{M}(m,n)$ - the $m \times n$ matrices with real entries, also denoted $\mathcal{M}$ when m and n are clear from context. For $A \in \mathcal{M}(m,n)$ we write $[A]_{ij}$ for the (i,j)th entry of A. If the entries are given as a_{ij}, we also write $A = (a_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$.
- $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ - the linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, also denoted $\mathcal{L}$ when $\mathbb{R}^n$ and $\mathbb{R}^m$ are clear from context.
- T_A for $A \in \mathcal{M}(m,n)$ - the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ induced by the standard action of an $m \times n$ matrix on an n-dimensional vector.
- $\mathcal{T}$ - the isomorphism from $\mathcal{M}(m,n)$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ defined by $A \mapsto T_A$.

5.2

- U - usually an open subset of $\mathbb{R}^n$.
- p - usually a point in U.
- x_j - the jth coordinate of a vector x
- $(Df)_p$ - the derivative of a function $f\colon U \to \mathbb{R}^m$ at a point $p \in U$
- $R(v)$ - the Taylor remainder (implicitly depending on a function $f\colon U \to \mathbb{R}^m$ and a point $p \in U$), defined by the formula $f(p+v) = f(p) + (Df)_p(v) + R(v)$
- f_i - the ith coordinate function of a vector valued function f
- $\frac{\partial f_i(p)}{\partial x_j}$ - the jth partial derivative of the ith coordinate function of a vector valued function f
- Df - the (total) derivative or Fréchet derivative of f, usually thought of as a map $Df\colon U \to \mathcal{L}(R^n, R^m)$.

5.3

- $D^r f$ - the rth derivative of f, thought of as a map $D^r f\colon U \to \mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m)$.
- $\mathcal{L}^r(\mathbb{R}^n, \mathbb{R}^m)$ - r-linear maps from $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{r \text{ times}}$ to $\mathbb{R}^m$, isomorphic to $\mathcal{L}(\mathbb{R}^{n^r}, \mathbb{R}^m)$.
- $f_k \rightrightarrows f$ - for a function f and a sequence of functions $(f_k)_{k\in\mathbb{N}}$, with $f_k, f\colon U \to \mathbb{R}^n$, this notation means the sequence converges uniformly to f on U.
- $\|f\|_r$ - the C^r norm for an function $f\colon U \to \mathbb{R}^m$ of class C^r, defined by

$$\|f\|_r = \max\left\{\sup_{p\in U} |f(p)|, \ldots, \sup_{p\in U} \|(D^r f)_p\|\right\}.$$

- $C^r(U, \mathbb{R}^m)$ - the set of C^r functions $f\colon U \to \mathbb{R}^m$ with $\|f\|_r < \infty$.

5.7

- R - a rectangle in $\mathbb{R}^2$, usually given by $R = [a,b] \times [c,d]$. Has area denoted $|R| = (b-a)(d-c)$.

Notation

- G - a grid on a rectangle $R = [a, b] \times [c, d]$ given by $G = P \times Q$ where
$$P = \{a = x_0 < x_1 < \cdots < x_m = b\}$$
is a partition of $[a.b]$ and
$$Q = \{c = y_0 < y_1 < \cdots < y_n = d\}$$
is a partition of $[c, d]$
- R_{ij} - a subrectangle of R determined by a grid G. If G is as above, then $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Note that $|R_{ij}| = (x_i - x_{i-1})(y_j - y_{j-1}) = \Delta x_i \Delta y_j$.
- S - sample points in R determeind by a grid G. If G is as above, then $S = \{s_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ where s_{ij} is **any** point in R_{ij}.
- $R(f, G, S)$ - the Riemann sum of f corresponding to the grid G and samples points S. It is the number
$$R(f, G, S) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(s_{ij})|R_{ij}|$$
- $L(f, G)$ - the lower (Darboux) sum for a bounded function f with respect to a grid G. It is the number
$$L(f, G) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}|R_{ij}|,$$
where $m_{ij} = \inf\{f(x, y) : (x, y) \in R_{ij}\}$.
- $U(f, G)$ - the upper (Darboux) sum for a bounded function f with respect to a grid G. It is the number
$$U(f, G) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij}|R_{ij}|,$$
where $M_{ij} = \sup\{f(x, y) : (x, y) \in R_{i,j}\}$.
- $\int_R f$ - the Riemann integral of the function f over the rectangle R
- $\underline{\int}_R f$, $\overline{\int}_R f$ - the lower and upper integrals of f, respectively, defined as the
$$\underline{\int}_R f = \sup_G L(f, G) \qquad \overline{\int}_R f = \inf_G U(f, G).$$
For bounded functions f, these always exist and $\underline{\int}_R f \leq \overline{\int}_R f$ with equality if and only if f is Riemann integrable.
- χ_S for a subset $S \subset \mathbb{R}^2$ - the characteristic function of S, defined by $\chi_S(x, y) = 1$ if $(x, y) \in S$ and $\chi_S(x, y) = 0$ otherwise.
- $|S|$ for a bounded subset $S \subset \mathbb{R}^2$ - the area of the set S, which exists if S is Riemann measurable. It is defined as $|S| = \int_R \chi_S$ for any rectangle $R \supset S$.
- $\int_S f$ - for a Riemann measurable set $S \subset \mathbb{R}^2$ and f Riemann integrable on some rectangle $R \supset S$, this integral is defined as $\int_R f\chi_S$
- $\text{Jac}_z(\varphi)$ - the Jacobian of C^1 function $\varphi : U \to \mathbb{R}^n$ (defined on an open subset $U \subset \mathbb{R}^n$) at a point z, given by the number $\det((D\varphi)_z)$.

5.8

- $C_k(\mathbb{R}^n)$ - the set of k-cells in $\mathbb{R}^n$, which are smooth functions $\varphi\colon [0,1]^k \to \mathbb{R}^n$. The unit k-cube $[0,1]^k$ may also be denoted I^k.
- $\frac{\partial \varphi_I}{\partial u}$ - for $\varphi \in C_k(\mathbb{R}^n)$ and $I = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ this is the partial Jacobian of φ defined at $u \in [0,1]^k$ by
$$\frac{\partial \varphi_I}{\partial u}(u) = \det \begin{bmatrix} \frac{\partial \varphi_{i_1}}{\partial u_1}(u) & \cdots & \frac{\partial \varphi_{i_1}}{\partial u_k}(u) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_{i_k}}{\partial u_1}(u) & \cdots & \frac{\partial \varphi_{i_k}}{\partial u_k}(u) \end{bmatrix}.$$
Also denoted $\frac{\partial \varphi_I}{\partial u} = \frac{\partial(\varphi_{i_1}, \ldots, \varphi_{i_k})}{\partial(u_1, \ldots, u_k)}$.
- dy_I - a basic differential k-form on $\mathbb{R}^n$. For $I = \{1, \ldots, n\}^k$ this is the functional on $C_k(\mathbb{R}^n)$ defined by
$$dy_I(\varphi) = \int_{[0,1]^k} \frac{\partial \varphi_I}{\partial u},$$
and this number is called the I-shadow area of φ.
- $f dy_I$ - a simple differential k-form on $\mathbb{R}^n$. For $I = \{1, \ldots, n\}^k$ and $f\colon \mathbb{R}^n \to \mathbb{R}$ smooth this is the functional on $C_k(\mathbb{R}^n)$ defined by
$$f dy_I(\varphi) = \int_{[0,1]^k} f \circ \varphi \frac{\partial \varphi_I}{\partial u}.$$
- $\Omega^k(\mathbb{R}^n)$ - the set of (general) differential k-forms on $\mathbb{R}^n$. These are linear combinations of simple k-forms: $\omega = \sum f_I dy_I$.
- $C^k(\mathbb{R}^n)$ - the set of all functionals on $C_k(\mathbb{R}^n)$; thus $C^k(\mathbb{R}^n) \supsetneq \Omega^k(\mathbb{R}^n)$.
- $\int_\varphi \omega$ - for $\varphi \in C_k(\mathbb{R}^n)$ and $\omega \in \Omega^k(\mathbb{R}^n)$ this is equivalent notation for $\omega(\varphi)$.
- $\alpha \wedge \beta$ - the wedge product of two forms α and β. For $\alpha = \sum_I a_I dy_I \in \Omega^k(\mathbb{R}^n)$ and $\beta = \sum_J b_J dy_J \in \Omega^\ell(\mathbb{R}^n)$, this is the $k+\ell$-form on $\mathbb{R}^n$ given by $\sum_{I,J} a_I b_J dy_{IJ}$.
- $d\omega$ - the exterior derivative of form ω. For $\omega = \sum f_I dy_I$, we have $d\omega = \sum d(f_I) \wedge dy_I$, where $d(f_I)$ is the 1-form given by
$$d(f_I) = \sum_{j=1}^n \frac{\partial f_I}{\partial y_j} dy_j.$$
- $T_*\varphi$ - the pushforward of $\varphi \in C_k(\mathbb{R}^n)$ by a smooth map $T\colon \mathbb{R}^n \to \mathbb{R}^m$. It is the k-cell in $\mathbb{R}^m$ given by $T_*\varphi := T \circ \varphi \in C_k(\mathbb{R}^m)$.
- $T^*\omega$ - the pullback of $\omega \in \Omega^k(\mathbb{R}^n)$ by a smooth map $T\colon \mathbb{R}^m \to \mathbb{R}^n$. It is the k-form on $\mathbb{R}^m$ given by $[T^*\omega](\varphi) = \omega(T_*\varphi)$. More explicitly, if $\omega = \sum f_I dy_I$ then
$$T^*\omega = \sum_I f_I \circ T dT_I,$$
where if $I = (i_1, \ldots, i_k)$ then $dT_I = (dT_{i_1}) \wedge \cdots \wedge (dT_{i_k})$.

5.9

- $\partial\varphi$ - the boundary of $\varphi \in C_k(\mathbb{R}^n)$. It is a $(k-1)$-chain in $\mathbb{R}^n$: a formal linear combination of $(k-1)$-cells. It is given by
$$\partial\varphi = \sum_{j=1}^{k} (-1)^{j+1} \left(\varphi \circ \iota^{j,1} - \varphi \circ \iota^{j,0}\right),$$
where $\iota^{j,1}, \iota^{j,0} \colon [0,1]^{k-1} \to [0,1]^k$ are the maps defined by
$$\iota^{j,1}(u_1, \ldots, u_{k-1}) := (u_1, \ldots, u_{j-1}, 1, u_j, \ldots, u_{k-1})$$
$$\iota^{j,0}(u_1, \ldots, u_{k-1}) := (u_1, \ldots, u_{j-1}, 0, u_j, \ldots, u_{k-1}).$$
- $\delta^j\varphi$ - the jth dipole of $\varphi \in C_k(\mathbb{R}^n)$ for $j = 1, \ldots, k$. It is the $(k-1)$-chain given by
$$\delta^j\varphi = \varphi \circ \iota^{j,1} - \varphi \circ \iota^{j,0}.$$
- $\Omega^k(U)$ - the set of differential k-forms on U, for $U \subset \mathbb{R}^n$ an open subset. These are linear combinations of simple k-forms on U: $f dy_I$ for $f \colon U \to \mathbb{R}$ a smooth function and I a k-tuple in $\{1, \ldots, n\}$.
- $C_k(U)$ - the set of k-cells in U, which are smooth maps $\varphi \colon [0,1]^k \to U$.
- $B^k(U)$ - the set of exact k-forms on U; that is, the $\omega \in \Omega^k(U)$ on U such that $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}(U)$.
- $Z^k(U)$ - the set of closed k-forms on U; that is, the k-forms $\omega \in \Omega^k(U)$ such that $d\omega = 0$.
- $H^k(U)$ - the kth de Rham cohomology group of U, which is the vector space quotient of $Z^k(U)/B^k(U)$.

6.1

- $|B|$ - the volume of a box $B \subset \mathbb{R}^n$.
- $m^*(A)$ - The outer measure of a set $A \subset \mathbb{R}^d$, which is defined as the quantity:
$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} |B_k| \colon \{B_k\}_{k\in\mathbb{N}} \text{ is a countable covering of } A \text{ by open boxes}\right\}.$$

6.2

- $\mathcal{M}(\mathbb{R}^d)$ - the collection of Lebesgue measurable subsets of $\mathbb{R}^d$, also denoted $\mathcal{M}$, which are those subsets $E \subset \mathbb{R}^d$ satisfying the *Carathéodory condition*:
$$m^*(X) = m^*(X \cap E) + m^*(X \cap E^c) \qquad \forall X \subset \mathbb{R}^d.$$
- $m(E)$ - the Lebesgue measure of a Lebesgue measurable set $E \in \mathcal{M}$, which is just its outer measure.
- G_δ - a class of set: we say G is a G_δ set if it is the countable (or finite) intersection of open sets.
- F_σ - a class of set: we say F is an F_σ set if it is the countable (or finite) union of closed sets.

Measurable Functions

- $\overline{\mathbb{R}}$ - the set $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, called the extended real line. By convention, we take $0 \cdot \infty = 0$.

- a.e. - "almost everywhere." We write this when a condition/property/equality holds except possibly on a measure set; e.g. $f = g$ a.e. means that the functions f and g agree except possibly on a measure zero set.

The Lebesgue Integral

- $\mathcal{L}^+(\mathbb{R}^d)$ - the space of Lebesgue measurable functions $f\colon \mathbb{R}^d \to [0,\infty]$. Also denoted $\mathcal{L}^+$.
- $\int f\ dm$ - the Lebesgue integral of a function f.

Integrating $\overline{\mathbb{R}}$-valued Functions

- $f_\pm$ - the positive and negative parts of a Lebesgue measurable function $f\colon \mathbb{R}^d \to \overline{R}$. Defined as

$$f_+ := f\chi_{f^{-1}([0,\infty])} \qquad\qquad f_- := -f\chi_{f^{-1}([-\infty,0)])}$$

- $L^1(\mathbb{R}^d, m)$ - the space of Lebesgue integrable functions $f\colon \mathbb{R}^d \to \overline{R}$. Also denoted $L^1(m)$. Later redefined to be the set of equivalence classes of such functions under the equivalence relation

$$f \sim g \qquad \Longleftrightarrow \qquad f = g \text{ a.e.}$$

- $\|f\|_1$ - the L^1-norm for a function $f \in L^1(m)$. Defined as

$$\|f\|_1 = \int |f|\ dm.$$