Math 105

A Second Course in Analysis

5 Multivariable Calculus

5.1 Linear Algebra

We begin by recalling some basic ideas from linear algebra. Recall that a vector space V is a set whose elements we can add together and multiply by scalars ($\lambda \in \mathbb{R}$). Such sets have a dimension determined by the size of (any) basis (a linearly independent, spanning set)

Ex ① For $n \in \mathbb{N}$, n-dimensional Euclidean space

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\} = \{ (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R} \}$$

is a vector space with pointwise operations:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \qquad \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

The standard basis for $\mathbb{R}^n$ is $\{e_1, e_2, \ldots, e_n\}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$\boxed{\text{so } v \in \mathbb{R}^n = \sum_{i=1}^n v_i e_i}$

Thus $\dim(\mathbb{R}^n) = |\{e_1, \ldots, e_n\}| = n$.

Ex ② For $n, m \in \mathbb{N}$ denote by $M(m,n)$ ($= M$) the $m \times n$ matrices with real entries. Is a vector space also with pointwise operations.

Notation: For $A \in M$ with (i,j)-entry a_{ij}, we write

$$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

and $[A]_{ij} = a_{ij}$.

$\mathcal{M}$ has ~~[illegible]~~ basis $\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$
where $[E_{ij}]_{k\ell} = \delta_{i=k}\, \delta_{j=\ell}$.

Thus $\dim(\mathcal{M}(m,n)) = m \cdot n$. In fact, $\mathcal{M}(m,n) \cong \mathbb{R}^{m\cdot n}$.

- Recall that every $\underset{=(a_{ij})}{A} \in \mathcal{M}(m,n)$ defines a linear transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{(warning: } n \to m\text{)}$$

by

$$T_A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix}$$

$$T_A\left(\sum_{j=1}^{n} x_j e_j\right) \qquad = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} x_j) e_i$$

In fact, denoting by $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ $(= \mathcal{L})$ the set of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ (another vector space),

$$\mathcal{T} : \mathcal{M} \to \mathcal{L}$$
$$A \mapsto T_A$$

defines an isomorphism of vector spaces. Hence we can always think of matrices as linear transformations and vice versa. (The former are better for computing, the latter for theory).

- Recall, that matrices also have a multiplication operation, for $A = (a_{ij}) \in \mathcal{M}(m,n)$, $B = (b_{ij}) \in \mathcal{M}(n,p)$ $AB \in \mathcal{M}(m,p)$ with

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Also, for linear transformations we have composition.

<u>Thm</u>: For $A \in \mathcal{M}(m,n)$ and $B \in \mathcal{M}(n,p)$

$$T_A \circ T_B = T_{AB}$$

Pf Exercise — check on the std basis for $\mathbb{R}^n$. $\square$

Recall that a norm on a vector space V is a map $|\cdot|: V \to \mathbb{R}$ satisfying:

(a) $\forall v \in V$, $|v| \geq 0$ with equality iff $v=0$

(b) $|\lambda v| = |\lambda| \cdot |v| \quad \forall \lambda \in \mathbb{R}$

(c) $|v+w| \leq |v| + |w|$

When V is equipped with a norm, we call it a normed space. If we need to specify the space on which the norm is defined, we may write $|\cdot|_V$.

Observe that:

$$d(v,w) = |v-w|$$

defines a metric for V.

Ex: $\mathbb{R}^n \ni \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} =: v$ has $|v| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Recall that in this case, the norm relates to the dot product (inner product):

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 y_1 + \cdots + x_n y_n$$

and $|v| = \langle v, v \rangle^{1/2}$.

Also observe that $x_j = \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, e_j \right\rangle$.

Def: Let V and W be normed spaces. For a linear transformation $T: V \to W$, its operator norm is the quantity:

$$\|T\| := \sup \left\{ \frac{|Tv|_W}{|v|_V} : v \in V \setminus \{0\} \right\}$$

We can think of $\|T\|$ has the maximum amount it stretches a vector. Consequently:

$$\|T \circ S\| \leq \|T\| \cdot \|S\|$$

For $T: V \to W$, $S: U \to V$. Exercise: check rigorously

Ex For $A \in \mathcal{M}$, can consider $\|T_A\|$.

Exercise: For $A \in \mathcal{M}(n,n)$, $A^T = A$, show
$$\|A\| = \max |\lambda_i|$$
where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A
(use problems 4 and 5 from HW 1)

Thm: Let $T: V \to W$ be a lin. trans between two normed spaces. The following are equivalent (TFAE):

(a) $\|T\| < \infty$;

(b) T is uniformly continuous

(c) T is continuous

1/17/2018 (d) T is continuous at the origin.

Pf: (a) $\Rightarrow$ (b) As T is linear, for $v, v' \in V$ we have:
$$|Tv - Tv'| = |T(v-v')| \leq \|T\| \, |v - v'|$$
Thus for $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{\|T\|}$ in def'n. of unif. cty.

(b) $\Rightarrow$ (c) Immediate.

(c) $\Rightarrow$ (d) Immediate.

(d) $\Rightarrow$ (a) Taking $\varepsilon = 1$, let $\delta > 0$ be s.t. if $u \in V$ satisfies $|u| = |u - 0| < \delta$, then
$$|Tu| = |Tu - 0| < 1.$$
Now, for arbitrary $v \in V$, set
$$\lambda = \frac{\delta}{2|v|}$$
and
$$u := \lambda v.$$
Then $|u| = \frac{\delta}{2|v|} \cdot |v| = \frac{\delta}{2} < \delta$.

Thus
$$\frac{|Tv|}{|v|} = \frac{|T \frac{1}{\lambda} u|}{|\frac{1}{\lambda} u|} = \frac{|Tu|}{|u|} < \frac{1}{|u|} = \frac{2}{\delta}$$
which implies $\|T\| \leq \frac{2}{\delta} < \infty$. $\square$

<u>Thm</u> Every lin. trans. $T: \mathbb{R}^n \to W$ is cts, and every isomorphism $T: \mathbb{R}^n \to W$ is a homeomorphism (inverse is cts)

<u>Pf</u>: Letting $\{e_1, \dots, e_n\}$ be the std basis for $\mathbb{R}^n$, set

$$M = \max\{ |Te_1|_W, \dots, |Te_n|_W \}.$$

For $v = \sum v_j e_j \in \mathbb{R}^n$, note that

$$|v_j| \le \sqrt{|v_j|^2} \le \sqrt{v_1^2 + \dots + v_n^2} = |v|$$

Thus

$$|Tv|_W = \left| \sum_{j=1}^n v_j T(e_j) \right|_W \le \sum_{j=1}^n |v_j| \, |T(e_j)|_W$$

$$\le \sum_{j=1}^n |v| \cdot M = n|v| \cdot M$$

Thus $\|T\| \le n \cdot M < \infty$. By the previous thm, T is ~~cts~~ cts.

Now, let $T: \mathbb{R}^n \to W$ be an isomorphism. Then by the above T is cts. It remains to show T^{-1} is cts. Consider the unit sphere

$$S^{n-1} := \{ u \in \mathbb{R}^n : |u| = 1 \}.$$

As a closed and bounded set, we know S^{n-1} is compact (Heine + Borel). Since T is cts, $T(S^{n-1})$ is compact. Since T is injective, and $0 \notin S^{n-1}$, $0_W = T(0) \notin T(S^{n-1})$.

~~Thus $\{0_W\}$ and $T(S^{n-1})$ are disjoint, compact sets in the normed (metric) space W.~~ Consequently

$$c := \inf\{ |Tu - 0_W| : u \in S^{n-1} \} > 0.$$

That is, $|Tu| = |Tu - 0| \ge c \quad \forall u \in S^{n-1}$.

For $v \in \mathbb{R}^n \setminus \{0\}$ write $v = \lambda u$ where

$$\lambda = |v| \quad \text{and} \quad u = \frac{v}{|v|}$$

linearity of T yields $Tv = \lambda Tu$ and so $|Tv| \ge |\lambda| \cdot c = |v| c$, that is, $\frac{|v|}{|Tv|} \le \frac{1}{c}$.

Hence, for $w \in W$ if $v = T^{-1}(w)$, then

$$\frac{|T^{-1}(w)|}{|w|} = \frac{|v|}{|Tv|} \leq C$$

$\Rightarrow \|T^{-1}\| \leq \frac{1}{c} < \infty$. The previous thm therefore implies T^{-1} is cts. $\square$

<u>Remark</u>: $\frac{1}{\|T^{-1}\|}$ is called the <u>conorm</u> of T and represents the smallest factor by which ~~T~~ shrinks a ~~f~~ vector

<u>Cor</u>: For finite dim'l normed spaces, all lin. trans. are cts. and isomorphisms are homeomorphisms. In particular, if a finite dim'l vector space has two norms, then the identity map yields a homeomorphism between the normed spaces. In particular, $T: M \to \mathcal{L}$ is a homeomorphism.

<u>Pf</u> Exercise. $\square$

5.2 Derivatives

Recall that for $U \subseteq \mathbb{R}$ open, ~~and~~ $f: U \to \mathbb{R}$ a function, and $x \in U$ ~~we~~ f has derivative $f'(x)$ ~~at~~ at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

we want to adapt this definition to ~~function~~ functions of the form

$$f: U \to \mathbb{R}^m$$

where $U \subseteq \mathbb{R}^n$ is open. However, if $h \in \mathbb{R}^n$ we have no notion of division. & also $f(x)$ $f(x) \cdot x$ for doesn't work $\frac{f(x+h) - f(x)}{|h|}$

Fortunately, we have the following alternative definition:

$$f(x+h) = f(x) + f'(x) \cdot h + R(h) \quad \xrightarrow{\text{where}} \quad \lim_{h \to 0} \frac{R(h)}{|h|} = 0$$