Def The C^r-norm of a C^r function $f: U \to \mathbb{R}^m$ is

$$\|f\|_r := \max\{\sup_{p\in U} |f(p)|, \dots, \sup_{p\in U} \|(D^r f)_p\|\}$$

~~We~~ denote

$$C^r(U, \mathbb{R}^m) := \{f: U \to \mathbb{R}^m \mid f \text{ is of class } C^r \text{ and } \|f\|_r < \infty\}.$$

Cor: $C^r(U, \mathbb{R}^m)$ with $\|\cdot\|_r$ is a complete normed space (i.e. a Banach space)

Pf: Exercise. $\square$

Prop (C^r M-test): For $(f_k)_{k\in\mathbb{N}} \subseteq C^r(U, \mathbb{R}^m)$, if ~~s.t.~~ $\exists$ constants $(M_k)_{k\in\mathbb{N}} \subseteq [0, +\infty)$ s.t. $\|f_k\|_r \le M_k$ for each $k \in \mathbb{N}$ and $\sum M_k$ is conv., then $(f_k)_{k\in\mathbb{N}}$ is unif. C^r-convergent to some f and

$$D^s f = \sum_{k=1}^{\infty} D^s f_k \qquad \forall\, s \le r.$$

5.4 Implicit and Inverse Functions

Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open and fix $f: U \to \mathbb{R}^m$.
We fix a point $(x_0, y_0) \in U$ and suppose $f(x_0, y_0) = z_0$.
Our goal is to show that (under certain) conditions, the equation

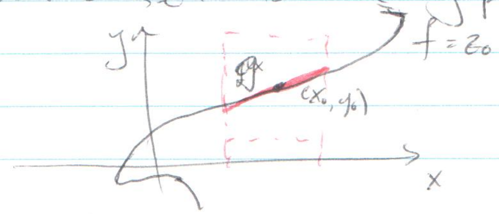
$$f(x, y) = z_0$$

has a solution set of points near (x_0, y_0) for which $y = g(x)$ for some function g. That is,

$$f(x, g(x)) = z_0$$

and the solution set is the graph of g

$n = m = 1$



Def The set $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : f(x,y) = z_0\}$ is called the z_0-locus of f.
The function g satisfying
$$f(x, g(x)) = z_0$$
for $(x, g(x))$ near (x_0, y_0), is called the implicit function defined by $f(x,y) = z_0$.

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Thm (Implicit Function Theorem)
Let $U \subseteq \mathbb{R}^n \oplus \mathbb{R}^m$ be open, let $f: U \to \mathbb{R}^m$ be of class C^r, $1 \le r \le \infty$, and let $(x_0, y_0) \in U$ satisfy $f(x_0, y_0) = z_0$. Consider $B \in M(m,m)$ with
$$[B]_{ij} = \frac{\partial f_i(x_0, y_0)}{\partial y_j}$$
If B is invertible, then near (x_0, y_0) the z_0-locus of f is the graph of a unique function $g: \mathbb{R}^n \to \mathbb{R}^m$. Moreover, g is of class C^r.

Pf: WLOG, we may assume $(x_0, y_0) = (0, 0)$ and $z_0 = 0$ (consider $g(x,y) = f(x+x_0, y+y_0) - z_0$). Then Taylor remainder of f at 0 is given by:
$$f(x,y) = 0 + (Df)_{(0,0)}(x,y) + R(x,y).$$
Note $(Df)_{(0,0)} \in \mathcal{L}(\mathbb{R}^{m+n}, \mathbb{R}^m) \cong \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \oplus \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ so we can write
$$(Df)_{(0,0)}(x,y) = Ax + By \quad (= (Df)_{(0,0)}(x,0) + (Df)_{(0,0)}(0,y))$$
with B as above and $A \in M(m,n)$
$$[A]_{ij} = \frac{\partial f_i(0,0)}{\partial x_j},$$
Now, we're trying to solve $f(x, g(x)) = 0$ for $g(x)$ which is equivalent to
$$A(x) + B(g(x)) + R^{(x,g(x))} = 0$$
$$\iff g(x) = -B^{-1}(Ax + R^{(x,g(x))})$$
If R doesn't depend on $y = g(x)$, then we're done since the above gives an explicit formula for g. This isn't true in general, but its dependence

on y is weak enough that may still obtain an implicit ~~formula~~ solution.

Consider, for $x\in\mathbb{R}^n$ the map $K_x : \mathbb{R}^m \to \mathbb{R}^m$

$$K_x : y \mapsto -B^{-1}(Ax + R(x,y))$$

If we can find y s.t. $K_x(y) = y$, then we are done. ~~Recall that fixed points~~ we require a lemma.

$\blacksquare$

<u>Lemma</u> (Contraction Mapping Principle / Banach Contraction Principle)

Let (E, d) be a complete metric space, and suppose for $f: E \to E$ $\exists\, c \in (0,1)$ s.t.

$$d(f(x), f(y)) \le c\, d(x,y) \qquad \forall x, y \in E.$$

Then f has a unique fixed point $p \in E$, and for any $x \in E$

$$\lim_{n\to\infty} \underbrace{f\circ f\circ \cdots \circ f}_{n \text{ times}}(x) = p.$$

<u>Pf</u>: Let $x \in E$, and define $x_0 = x$ and

$$x_n := \underbrace{f\circ f\circ \cdots \circ f}_{n \text{ times}}(x) \qquad n\in\mathbb{N}$$

We'll show $(x_n)_{n\in\mathbb{N}}$ is Cauchy. First observe that for $n \ge 1$

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \le c\, d(x_n, x_{n-1}) \le \cdots \le c^n d(x_1, x_0)$$

Thus for $m < n$ we have

$$d(x_n, x_m) \le \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \le \sum_{k=m}^{n-1} c^k d(x_1, x_0)$$

$$\le d(x_1, x_0) \sum_{k=m}^{\infty} c^k$$

$$= d(x_1, x_0) \frac{c^m}{1-c}$$

So for $\varepsilon > 0$, if $N \in \mathbb{N}$ is s.t.

$$d(x_1, x_0) \frac{c^N}{1-c} < \varepsilon$$

then $\forall n, m \ge N$, we have $d(x_n, x_m) < \varepsilon$. Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and therefore converges to some $p \in E$; we claim p is a fixed point of f. Indeed, f is cts, so

$$d(p, f(p)) = \lim_{n\to\infty} d(x_n, f(x_n))$$

$$= \lim_{n\to\infty} d(x_n, x_{n+1})$$

$$\leq \lim_{n\to\infty} c^n d(x_1, x_0) = 0.$$

Now, if $p' \in E$ is another fixed point, we have

$$d(p, p') = d(f(p), f(p')) \leq c \cdot d(p, p') < d(p, p'),$$

a contradiction unless $p = p'$. E

Finally, for any $y \in E$, the same argument as above shows $f\circ\cdots\circ f(y)$ converges to a fixed pt, but p is the unique fixed pt, so we have $\lim_{n\to\infty} \underbrace{f\circ f\circ\cdots\circ f}_{n}(y) = p$ $\square$

Application: Map on the floor.

Our goal is to show K_x maps a region near $0 \in \mathbb{R}^m$ to itself and is a contraction.

Observe that

$$R(x,y) = \cancel{Ax+By} + f(x,y) - Ax - By$$

is of class C^r. In particular

$$(DR)_{(0,0)} = \cancel{A} (Df)_{(0,0)} - [A + B] = 0$$

Thus, for small $r > 0$, if $|x|, |y| \leq r$ then

$$\left\| \frac{\partial R(x,y)}{\partial y} \right\| \leq \frac{1}{2\|B^{-1}\|}$$

where $\frac{\partial R(x,y)}{\partial y} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is defined by:

$$\frac{\partial R(x,y)}{\partial y}(w) = (DR)_{(x,y)}(0, w).$$

The C^1 MVT then implies for $|x|, |y_1|, |y_2| \leq r$

$$|K_x(y_1) - K_x(y_2)| = |B^{-1}(R(x,y_2) - R(x,y_1))|$$

$$\leq \|B^{-1}\| \cdot \left\| \int_0^1 \frac{\partial R}{\partial y}(x, y_1 + t(y_2-y_1))\, dt \right\| \cdot |y_2 - y_1|$$

$$\leq \|B^{-1}\| \cdot \frac{1}{2\|B^{-1}\|} |y_2 - y_1| \leq \frac{1}{2}|y_2 - y_1|$$

Also observe for $0<s\le r$ suff. small, if $|x|\le s$, then

$$|K_x(0)| = |-B^{-1}(Ax + R(x,0))| \le \frac{r}{2}$$

Hence for $|x|\le s$, $|y|\le r$

$$|K_x(y)| \le |K_x(y) - K_x(0)| + |K_x(0)| \le \frac{1}{2}|y-0| + \frac{r}{2} \le \frac{r}{2} + \frac{r}{2} = r.$$

That is, for $|x|\le s$, K_x maps

$$E = \{y \in \mathbb{R}^m : |y| \le r\}$$

2/7/2018 to itself and is a contraction. The contraction mapping theorem therefore implies that K_x has a unique fixed point $y\in E$, which we denote $g(x)$.

In summary, for $|x|\le s$ ~~$\exists!\, g \in \mathbb{R}^m$ s.t. $|y|\le r$~~ and $|g(x)|\le r$ and $f(x,g(x)) = 0$.

It remains to show that g is of class C^r. We first show it is Lipschitz at 0. Indeed For $|x| << s$ we have $|g(x)|\le r$ and

$$\begin{aligned}|g(x)| &= |K_x(g(x))| \le |K_x(g(x)) - K_x(0)| + |K_x(0)| \\ &\le \frac{1}{2}|g(x)-0| + |B^{-1}(Ax + R(x,0))| \\ &= \frac{1}{2}|g(x)| + |B^{-1}(Ax + R(x,0))| \\ &\le \frac{1}{2}|g(x)| + 2\|B^{-1}\|\cdot(\|A\|+1)|x|\end{aligned}$$

using R sublinear

$$\Leftrightarrow \quad |g(x)| \le 4\|B^{-1}\|\cdot(\|A\|+1)|x|.$$

Let $L := 4\|B^{-1}\|\cdot(\|A\|+1)$. Note that this also implies g is cts at 0.

By the chain rule, if $(Dg)_0$ exists, we must have:

$$Bg(x) = -(Ax + R(x,g(x))) \Rightarrow B(Dg)_0 = -A - (DR)_{(0,0)}\begin{pmatrix} I \\ (Dg)_0\end{pmatrix}$$

Also $f(x,g(x))$ constant: $0 = D(f(x,g(x)))_x = (Df)_{(x,g(x))}\begin{pmatrix} I \\ (Dg)_x\end{pmatrix}$

in which case $(Dg)_0 = -B^{-1}A$. Thus we'll show this by computing the Taylor remainder.

$$|g(x) - g(0) - (-B^{-1}Ax)| = |K_x(g(x)) - 0 + B^{-1}Ax|$$
$$= |-B^{-1}(Ax + R(x,g(x))) + B^{-1}Ax|$$
$$= |B^{-1}R(x,g(x))|$$
$$\le \|B^{-1}\| \cdot |R(x,g(x))|$$
$$\le \|B^{-1}\| \frac{|R(x,g(x))|}{|(x,g(x))|} \cdot (|x| + |g(x)|)$$
$$\le \|B^{-1}\| \cdot e(x,g(x)) (1+L)|x|$$

~~Thus this used~~ Since $g(x) \to 0$ as $x \to 0$, $e(x,g(x)) \to 0$ as $x \to 0$ and hence the above is sublinear. That is, $(Dg)_0$ exists and is $-B(A^{-1}$.

The same proof ~~[illegible]~~ can be used to show g is diff'ble at x near 0 (we only used estimates known to hold near 0), with $(Dg)_x = -B_x^{-1} \circ A_x$ where

$$A_x = \frac{\partial f(x,g(x))}{\partial x} \qquad B_x = \frac{\partial f(x,g(x))}{\partial y}$$

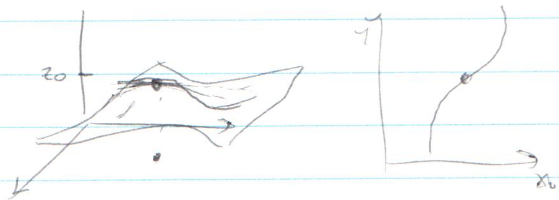
~~(Note B_x is invertible for x (and hence $g(x)$) near 0, since $\det(B_0) \neq 0$ and $x \mapsto \det(B_x)$ is cts).~~

Since g is cts (it is diff'ble in fact), and f is C^1, A_x + B_x are cts. This implies B_x is invertible for x (and hence $g(x)$) near 0 since $x \mapsto \det(B_x)$ is cts & $\det(B_0) \neq 0$. Moreover, $x \mapsto B_x^{-1}$ is cts by Cramer's rule for computing inverses. Hence $x \mapsto (Dg)_x$ is cts and g is of class C^1.

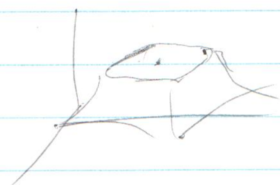
To complete the proof, we proceed by induction: for $2 \le r < \infty$ assume we shown the theorem is true for $r-1$. So if f is of class C^r, we have that g is C^{r-1}. ~~By~~ ~~Cramer's rule~~ As compositions of C^{r-1} functions, A_x + B_x are C^{r-1}, and then so is B_x^{-1}

by Cramer's rule. Hence $(Dg)_x = -B_x^{-1} \circ A_x$ is C^{r-1} $\Rightarrow$ g is C^r. If f is C^∞, then this shows g is C^r for all $r \geq 1$, hence is C^∞. $\square$

Ex: (1) $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$



(2)



$\frac{\partial f}{\partial y}(x_0,y_0) = 0$

(3) $f: \mathbb{R}^m \to \mathbb{R}^m$ (i.e. $n = 0$)
$\rightsquigarrow$ Inverse Function theorem.

2/9/2018

Def: For $U \subseteq \mathbb{R}^m$, A function $f: U \to \mathbb{R}^n$ is called a C^r-diffeomorphism if it is a C^r bijection from U to $f(U)$ whose inverse $f^{-1}: f(U) \to \mathbb{R}^m$ is also C^r.

Ex (1) Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$
Then $f: \mathbb{R}^3 \to \mathbb{R}^3$
$f(x,y,z) = \cancel{a(x-x_0)^2 + b(y-y_0)^2 +} (ax+x_0, by+y_0, (z+z_0))$
is a C^∞ diffeomorphism to takes S^2 to the ellipsoid ~~defi~~ given by $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$

(2): $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a bijection, but not a diffeomorphism.

Remark: If you care about the shape of surfaces, this is the right type of isomorphism to consider. Can't tell the difference when standing on one or the other.
Note: S^2 not diffeomorphic to a disc.

$(q-q)+q'$
$(q-y)+q'$

<u>Thm (Inverse Function Theorem)</u>
Let $U \subseteq \mathbb{R}^m$ be open, and let $f: U \to \mathbb{R}^n$ be of class C^r $1 \le r \le \infty$. If for some $p \in U$, $(Df)_p$ is invertible, then near p f is a C^r diffeomorphism.

<u>Pf:</u> ~~Def~~ Define $F: U \times \mathbb{R}^m \to \mathbb{R}^m$ by
$$F(x,y) = f(x) - y$$
Set $q = f(p)$. Clearly F is C^r, ~~and~~ $F(p,q) = f(p) - q = 0$, and the matrix
$$\left(\frac{\partial F_i(p,q)}{\partial x_j}\right)_{1 \le i,j \le m} \text{ is } (Df)_p.$$
Since $(Df)_p$ is invertible, the Implicit Function theorem (with x & y interchanged) implies there are nbhds U_p of p and V_q of q and a C^r implicit function $h: V_q \to U_p$ uniquely def. by
$$0 = F(h(y), y) = f(h(y)) - y$$
That is, for $y \in V_q$ $f(h(y)) = y$. Observe that ~~$h(q) = p \in U$~~ since $F(p,q) = 0$ and h is unique, $h(q) = p$.
Then
$$I = D(f \circ h)_q = (Df)_p \circ (Dh)_q$$
Hence $(Dh)_q$ is invertible. We claim $h \circ f = id$ near p as well, in which case f is a diffeo.
Indeed, apply the above analysis to h to obtain nbhds $V_q' \subseteq V_q$ and $U_p' \subseteq U_p$ and a C^r implicit function $g: U_p' \to V_q'$ s.t., by the above $h \circ g = id$. But then on U_p'
$$f = f \circ (h \circ g) = (f \circ h) \circ g = g$$
Thus $h \circ f = h \circ g = id$ on U_p'. So h is a local left & right inverse for $f \Rightarrow f$ is a local diffeomorphism $\square$

Skipping 5.5 & 5.6 - read.